

Investment Timing under Incomplete Information

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Abstract

We study the decision of when to invest in a project whose value is perfectly observable but driven by a parameter that is unknown to the decision-maker ex-ante. This problem is equivalent to an optimal stopping problem for a bivariate Markov process. Using filtering and martingale techniques, we show that the optimal investment region is characterized by a continuous and non-decreasing boundary in the value-belief state space. This generates path-dependency in the optimal investment strategy. We further show that the decision-maker always benefit from an uncertain drift relative to an average drift situation, and that the value of the option to invest is not globally increasing with respect to the volatility of the value process.

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1. Introduction. Uncertainty and irreversibility have long been recognized as key determinants of investment (Arrow and Fisher 1974, Henry 1974). In contrast with the net present value rule, which implicitly requires investment expenditures to be fully recoverable, or investment opportunities to be seized on a now-or-never basis, the real option literature has emphasized the ability of firms to delay their irreversible investment decisions (Dixit and Pindyck 1994). In the presence of sunk costs, this flexibility in the timing of investment is valuable because it gives firms the option to wait for new information. Conversely, the loss of this option value at the time the firm invests generates an additional opportunity cost of investment. As a consequence, investment options are exercised significantly above the point at which expected discounted cash-flows cover the sunk investment expenditures.

In the benchmark case of a single project, the optimal investment policy can be formally obtained as the solution to an optimal stopping problem. The prototype of this approach is the model of McDonald and Siegel (1986), in which the underlying value of the project evolves as a geometric Brownian motion. Under this assumption, the optimal investment time can be explicitly characterized using Samuelson and McKean's (1965) celebrated smooth-fit principle. These results have been recently extended in various directions. For instance, Hu and Øksendal (1998) study an environment in which the investment cost is driven by a sum of correlated geometric Brownian motions, while Mordecki (1999) considers the case of a jump-diffusion value process. However, a common feature of these contributions is their focus on complete information settings, in which decision-makers have no uncertainty about the fundamental characteristics of investment projects.

In this paper, we consider the problem of investing in a project whose value is observable but driven by a parameter that is unknown to the decision-maker *ex-ante*. Specifically, the value of the project evolves as an arithmetic Brownian motion, whose drift can take one of two non-negative values. (Extensions to arbitrary finite numbers of values for the drift, including negative ones, and to geometric Brownian motion, will be considered in Section 7.) Uncertainty about the drift adds a structural risk component besides the standard diffusion component of the value process. This captures in a simple way a variety of empirically relevant investment situations. For instance, a firm might not know the exact growth characteristics of a market on which it contemplates investing. Alternatively, the owner of an asset who considers selling it might not know the future distribution of potential buyers' valuations. By observing the evolution of the value, the decision-maker can update his beliefs about the unknown drift of the value process. This information is noisy, however, since it does not allow to distinguish perfectly between the relative contributions of the drift and diffusion components to the instantaneous variations of the value.

The filtering techniques of Liptser and Shiryaev (1977) allow us to restate our problem recursively as an optimal stopping problem for a bivariate Markov process. The relevant state variables are the current value of the project and the posterior belief of the decision-maker about the unknown drift of the value process. The multi-dimensionality of the state space is a key feature of our model, that distinguishes it from related investment or learning models. It reflects the fact that the value process coincides with the observation process, so that the diffusion component is both a genuine component of the value and a source of noise for the identification of the drift. While this prevents us from using smooth-fit techniques to characterize the optimal investment strategy, the martingale approach developed by Lakner (1995), Karatzas (1997), Karatzas and Zhao (2001) or Beneš et al. (2004) in the context of optimal control problems with partial observation allows us to derive analytical properties of the value of the investment option as a function of the current state variables.

An important analytical result is that the optimal investment region is characterized by

a continuous and non-decreasing boundary in the value-belief state space. Thus, in contrast with standard real options models, the optimal decision rule is not described by a simple threshold for the current value of the investment, above which it becomes optimal to invest no matter the past evolution of value. The presence of learning generates this path-dependency, although there is no option to suspend or abandon the project, in contrast to Dixit (1989). As a result, a striking feature of the optimal investment strategy is that it may be rational to invest after a drop in the value of the investment. This is because such a drop brings bad news about the unknown drift, and thus about the future evolution of the value, thereby reducing the current opportunity cost of investment. If the current value of the investment is high enough relative to his new estimate of the drift, the decision-maker may give up on learning, reflecting that “a bird in the hand is worth two in the bush.”

An important question is whether uncertainty about the drift actually benefits or harms the decision-maker. Specifically, we compare the value of an investment project with unknown drift and learning with that of an investment project with same volatility in which the drift of the value process is known and equal to the prior expectation of the drift in the first project. Using dynamic programming techniques, we show that the decision-maker always prefers the former unknown drift project to the latter average drift project, despite the fact that the value process only conveys a noisy signal of the drift. In other terms, an investment opportunity with uncertain growth prospects dominates one with average growth prospects. The intuition of this result is particularly easy to grasp whenever the volatility of the observation-value process is close to zero. In these circumstances, the value becomes a very accurate signal of the drift, so that learning occurs at a very fast rate. The fact that the unknown drift problem has a higher value for the decision-maker than the average drift problem then simply reflects that the value of the latter is convex with respect to the drift.

To get some intuition about the shape of the investment boundary, as well as about the wedge between the unknown drift problem and the average drift problem, we perform a local analysis whenever the volatility of the observation-value process is close to zero. We show that, as the volatility converges to zero, the loss in value arising from the need to learn about the drift vanishes, so that the value of the incomplete information problem converges to that of the complete information problem. An interesting by-product of this analysis is that, in contrast with standard models of investment under uncertainty, the value of the unknown drift problem is not everywhere increasing with respect to the volatility. This illustrates again the duality of the value process in our model. An increase in the volatility makes the payoff of the decision-maker upon investing more volatile, which tends to increase the value of the investment opportunity and the incentive to delay investment. However, since the value process coincides with the observation process, this also lowers the rate at which the decision-maker accumulates information about the unknown drift, which tends to decrease the value of the investment opportunity and the incentive to delay investment.

The paper is organized as follows. The model is described in Section 2. In Section 3, we provide the recursive formulation of our problem, and derive some basic properties of the value function. Section 4 derives the continuity of the investment boundary. In Section 5, we compare the unknown drift problem with the average drift problem. Section 6 is devoted to a local study of the investment boundary whenever the volatility of the observation-value process is close to zero. In Section 7, we investigate extensions of our model to an arbitrary finite number of values for the drift, including negative ones, and to a geometric Brownian motion specification of the value process. Section 8 concludes. Proofs not given in the text are in the Appendix.

2. The investment problem.

2.1. The basic model. The following notation will be maintained throughout the paper. Time is continuous, and labelled by $t \geq 0$. Uncertainty is modelled by a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the precise description of which we defer until the end of this section. We denote by \mathbb{E} the expectation operator associated to \mathbb{P} . For any stochastic process $X = \{X_t; t \geq 0\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{F}^X = \{\mathcal{F}_t^X; t \geq 0\}$ is the \mathbb{P} -augmentation of the filtration $\{\sigma(X_s; s \leq t); t \geq 0\}$ generated by X , and \mathcal{T}^X is the set of \mathcal{F}^X -adapted stopping times, with values in $\mathbb{R}_+ \cup \{\infty\}$.

An infinitely lived decision-maker has to choose when to invest in a risky project. The investment is irreversible and entails a sunk cost $I > 0$. The value of the project follows an arithmetic Brownian motion with unknown drift μ and known variance σ ,

$$(1) \quad dV_t = \mu dt + \sigma dW_t; \quad t \geq 0,$$

where (W, \mathcal{F}^W) is a standard Wiener process independent of μ . In the basic version of the model, μ can take only two non-negative values, normalized to 0 and 1. We denote by v the initial value of the project. The decision-maker is risk-neutral, and discounts future revenues and costs at a constant rate $r > 0$.

A key assumption of our model is that the decision-maker does not know the true value of μ ex-ante. We denote by $p = \mathbb{P}[\mu = 1]$ his prior belief that $\mu = 1$. The decision-maker then perfectly observes the value process V , but neither the drift μ , nor the evolution of W . His information at any time t is thus summarized by \mathcal{F}_t^V . It is clear from (1) that the value process conveys some information about μ . However, because of the unobservable shocks σW to the value, this information is noisy.

At any time t prior to investment, the decision-maker chooses whether to pay the sunk cost I to earn the gross profit V_t , or to delay further his investment. Since the only information available to him ex-post is generated by the value process, his decision problem is to find a stopping time $\tau^* \in \mathcal{T}^V$ such that:

$$(2) \quad \sup_{\tau \in \mathcal{T}^V} \mathbb{E}[e^{-r\tau}(V_\tau - I)] = \mathbb{E}[e^{-r\tau^*}(V_{\tau^*} - I)].$$

We shall denote this problem by \mathcal{P} . Our objective is to characterize as fully as we can the optimal investment strategy for \mathcal{P} .

Our notation and modelling assumptions are worthy of some comments. First, it should be noted that we allow for non \mathbb{P} -almost surely finite stopping times. While this added degree of freedom is not useful when μ only takes non-negative values, it is required when the model is extended to allow for negative values of μ , see Section 7. The notation in (2) is without ambiguity since $e^{-r\tau}(V_\tau - I) = \lim_{t \rightarrow \infty} e^{-rt}(V_t - I) = 0$ \mathbb{P} -almost surely on $\{\tau = \infty\}$ for any $\tau \in \mathcal{T}^V$. Second, under the formulation (1), the value V of the project can become negative. This can arise if V represents the difference between the true value of the project and opportunity costs of investment not captured by I . One way to avoid this issue altogether is to assume that V follows a geometric Brownian motion with drift μ . We defer until Section 7 the extension of the model to this case.

For further reference, it will be useful to describe precisely the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Specifically, define $\Omega = \{0, 1\} \times C(\mathbb{R}_+)$ and $\mathcal{F} = 2^{\{0,1\}} \otimes \mathcal{B}(C(\mathbb{R}_+))$, where $C(\mathbb{R}_+)$ is the space of continuous real-valued functions on \mathbb{R}_+ with Borel σ -field $\mathcal{B}(C(\mathbb{R}_+))$. Finally, define $\mathbb{P} = \mathbb{M} \otimes \mathbb{W}$, where \mathbb{M} is a probability measure on $(\{0, 1\}, 2^{\{0,1\}})$ such that

$\mathbb{M}[\{1\}] = p$, and \mathbb{W} is the Wiener measure on $(C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+)))$. Now μ can be defined as a random variable on $(\{0, 1\}, 2^{\{0, 1\}})$ with distribution \mathbb{M} , and W as the coordinate mapping process on $(C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+)))$, $W_t(\phi) = \phi(t)$, $t \geq 0$, $\phi \in C(\mathbb{R}_+)$, which is a standard one-dimensional Brownian motion under the Wiener measure \mathbb{W} . Last, two filtrations on $(\Omega, \mathcal{F}, \mathbb{P})$ are worth emphasizing. (i) One is the filtration generated by μ and W , $\mathcal{F}^{\mu, W} = \{\sigma(\mu) \otimes \mathcal{F}_t^W; t \geq 0\}$, which corresponds to the information available to the modeller. Since μ and W are independent by construction, W is also a Brownian motion under this filtration. (ii) The other is the filtration generated by V , \mathcal{F}^V , which corresponds to the information available to the decision-maker. In contrast with the previous case, it should be noted that W is not a Brownian motion under this filtration.

2.2. Related literature. Before we proceed with the analysis, it might be helpful to briefly contrast our model with some closely related contributions.

As in the real option literature, we model the investment decision as an optimal stopping problem (McDonald and Siegel 1986, Dixit and Pindyck 1994). The distinguishing feature of our setting is that, since the decision-maker can observe neither the drift μ nor the evolution of the diffusion term W , he has incomplete information about the dynamics of the value process V . As a result, his beliefs about the drift become sensitive to information about the past evolution of the value, which in turn affects his anticipations about the future evolution of the value. This implies that the current value of the project is not a sufficient statistic for the investment problem \mathcal{P} .

The problem of sequentially testing the two alternative hypotheses on the drift μ given the observation process V has been solved by Chernoff (1972, §17.5) assuming constant linear waiting costs. The solution consists to accumulate information until an upper or lower threshold is reached by the process of posterior beliefs about the drift. By contrast, a key feature of our model is that the observation process coincides with the value process, so that the diffusion term W affects directly the payoff of the decision-maker. This implies that the current belief about the drift of the observation process is not a sufficient statistic for the investment problem \mathcal{P} .

Sequential control problems under incomplete information and learning have been the focus of the economics literature on optimal experimentation. Most contributions follow Chernoff (1972) as we do in parameterizing the unknown state of nature by the drift of the observation-value process. However, the current belief about the drift remains a sufficient statistic, because the value process is interpreted in a cumulative sense (Jovanovic 1979, Felli and Harris 1996, Bolton and Harris 1999, Keller and Rady 1999, Bergemann and Välimäki 2000, Bernardo and Chowdhry 2002), or because terminal payoffs depend only on beliefs (Moscarini and Smith 2001).

The classical consumption-portfolio problem (Merton 1971) has been recently extended to situations in which the investor is uncertain about the drift of the stock price process. While Lakner (1995) tackles the problem via martingale methods, Karatzas and Zhao (2001) show that the Bellman principle can be applied and leads to explicit solutions for particular choices of utility functions (see also Brennan 1998). In order to explain excess volatility and time-variability of stock returns, Veronesi (1999, 2000) studies rational expectation models of equilibrium asset prices in which the drift of fundamentals may shift between several unobservable states at random times.

3. Markov formulation. In this section, we first derive a recursive representation of \mathcal{P} , for which a natural state variable is the pair (V, P) formed by the current value of

investing in the project and the current belief about the drift of the value process. This allows us to prove the existence of an optimal stopping time for \mathcal{P} . We then use a change of measure argument to derive some basic properties of the value function.

3.1. An existence result. The decision-maker in \mathcal{P} faces a standard signal extraction problem. As μ is either equal to 0 or 1, his belief about μ at any time t is summarized by the probability that $\mu = 1$ conditional on information available up to time t , $P_t = \mathbb{P}[\mu = 1 \mid \mathcal{F}_t^V]$. From Liptser and Shiryaev (1977, Theorems 7.12 and 9.1), the belief process (P, \mathcal{F}^V) satisfies the following filtering equation:

$$(3) \quad dP_t = \frac{P_t(1 - P_t)}{\sigma} d\bar{W}_t; \quad t \geq 0,$$

where the innovation process (\bar{W}, \mathcal{F}^V) is a standard Wiener process given by:

$$(4) \quad d\bar{W}_t = \frac{dV_t - P_t dt}{\sigma}; \quad t \geq 0.$$

Heuristically, the change in beliefs dP_t is normally distributed with mean 0 and variance $[P_t^2(1 - P_t)^2/\sigma^2]dt$. By construction, (P, \mathcal{F}^V) is a martingale.

It is worth noting that 0 and 1 are absorbing barriers for the belief process. However, if the latter does not start at one of these values, then neither can be attained in finite time. These useful properties are summarized in the following lemma.

LEMMA 1. *For any $p \in [0, 1]$, the unique solution (P^p, \mathcal{F}^V) of (3) satisfying $P_0^p = p$ lies \mathbb{P} -almost surely in $[0, 1]$ and satisfies:*

- (i) $P^0 = 0$ and $P^1 = 1$, \mathbb{P} -almost surely;
- (ii) For any $p \in (0, 1)$, $\inf\{t \geq 0 \mid P_t^p \notin (0, 1)\} = \infty$, \mathbb{P} -almost surely.

Solving \mathcal{P} when p is equal to 0 or 1 amounts to finding an optimal stopping time for a discounted arithmetic Brownian motion with known drift 0 or 1. This is a problem for which a closed-form solution is available, and the optimal stopping time is a trigger strategy. When $p = 0$ or $p = 1$, the optimal investment strategy consists to invest when the value respectively reaches $v^*(0) = I + \sigma^2/\sqrt{2r\sigma^2}$ and $v^*(1) = I + \sigma^2/(\sqrt{1 + 2r\sigma^2} - 1)$, see Section 5. Note in particular that $v^*(1) > v^*(0) > I$, reflecting the option value of waiting.

Since W is not a Brownian motion under the filtration \mathcal{F}^V of the observation process, the formulation (1)–(2) of \mathcal{P} is not recursive. What the filtering approach allows us to do is to transform \mathcal{P} into a stopping time problem for a two-dimensional Markov process. Indeed, it is immediate from (3)–(4) that the joint value-belief process can be rewritten as:

$$(5) \quad d \begin{bmatrix} V_t^v \\ P_t^p \end{bmatrix} = \begin{bmatrix} P_t^p \\ 0 \end{bmatrix} dt + \begin{bmatrix} \sigma \\ P_t^p(1 - P_t^p)/\sigma \end{bmatrix} d\bar{W}_t; \quad t \geq 0,$$

where we have made explicit the dependence of V and P upon their respective initial values v and p . Since the innovation process \bar{W} , unlike W , is a Brownian motion under \mathcal{F}^V , it is clear from (5) that the value-belief process $X^{v,p} = (V^v, P^p)$ is a Markov process under \mathcal{F}^V . We can thus restate \mathcal{P} as the problem of finding a value function $G^* : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ and a stopping time $\tau^* \in \mathcal{T}^V$ such that:

$$(6) \quad G^*(v, p) = \sup_{\tau \in \mathcal{T}^V} \mathbb{E}[e^{-r\tau} g(X_\tau^{v,p})] = \mathbb{E}[e^{-r\tau^*} g(X_{\tau^*}^{v,p})]; \quad (v, p) \in \mathbb{R} \times [0, 1],$$

where $g(v, p) = v - I$ for any $(v, p) \in \mathbb{R} \times [0, 1]$. Given the recursive representation (6), we are in position to prove the existence of a solution to \mathcal{P} by using standard results in optimal stopping theory for Markov processes.

PROPOSITION 1.

- (i) *There exist a value function G^* and a \mathbb{P} -almost surely finite stopping time $\tau^* \in \mathcal{T}^V$ that solve (6);*
- (ii) *The coincidence set $\mathcal{S}^* = \{(v, p) \in \mathbb{R} \times [0, 1] \mid G^*(v, p) = g(v, p)\}$ for (6) is non-empty and satisfies $\mathcal{S}^* = \bigcup_{p \in [0, 1]} \{p\} \times [v^*(p), \infty)$, where $v^*(p) = \inf\{v \in \mathbb{R} \mid (v, p) \in \mathcal{S}^*\} < \infty$ for any $p \in [0, 1]$.*

Proposition 1 implies that $\tau^* = \inf\{t \geq 0 \mid (V_t^v, P_t^p) \in \mathcal{S}^*\} = \inf\{t \geq 0 \mid V_t^v \geq v^*(P_t^p)\}$, so that the optimal investment strategy consists to invest whenever the value-belief process crosses a boundary that depends on the current belief about μ . We will refer to v^* as the investment boundary function for the unknown drift problem.

3.2. A Girsanov transformation. Following Lackner (1995), Karatzas (1997) and Karatzas and Zhao (2001), we now construct a probability measure \mathbb{Q} under which V and μ are independent. Specifically, consider:

$$(7) \quad Z_t(\mu) = \exp\left(-\left(\frac{\mu}{\sigma}\right)W_t - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 t\right); \quad t \geq 0.$$

By construction, $(Z(\mu), \mathcal{F}^{\mu, W})$ is a martingale and $\mathbb{E}[Z_t(\mu)] = 1$ for each $t \geq 0$. For any such t , let \mathbb{Q}_t be the unique probability measure on $(\Omega, \mathcal{F}_t^{\mu, W})$ such that, for any $(A, B) \in \sigma(\mu) \times \mathcal{F}_t^W$, $\mathbb{Q}_t[A \times B] = \mathbb{E}[1_{A \times B} Z_t(\mu)] = \int_A \mathbb{Q}_t^m[B] \mathbb{M}(dm)$, where, for any $m \in \{0, 1\}$:

$$\mathbb{Q}_t^m[B] = \mathbb{E}_{\mathbb{W}}\left[1_B \exp\left(-\left(\frac{m}{\sigma}\right)W_t - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 t\right)\right]; \quad B \in \mathcal{F}_t^W.$$

Here, $\mathbb{E}_{\mathbb{W}}$ is the expectation operator associated to the Wiener measure \mathbb{W} . It follows from a standard extension argument that, for any $m \in \{0, 1\}$, there exists a unique probability measure \mathbb{Q}^m on $(C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+)))$ such that $\mathbb{Q}^m[B] = \mathbb{Q}_t^m[B]$ for any $t \geq 0$ and $B \in \mathcal{F}_t^W$ (Karatzas and Shreve 1991, §3.5.A, Discussion). To complete the construction, define \mathbb{Q} as the unique probability measure on (Ω, \mathcal{F}) such that $\mathbb{Q}[A \times B] = \int_A \mathbb{Q}^m[B] \mathbb{M}(dm)$ for any $(A, B) \in \sigma(\mu) \times \mathcal{B}(C(\mathbb{R}_+))$.

By construction, $\mathbb{Q}[C] = \mathbb{E}[1_C Z_t(\mu)]$ for any $C \in \mathcal{F}_t^{\mu, W}$. As $\{\mu = 1\} \in \mathcal{F}_0^{\mu, W}$ and $Z_0(\mu) = 1$, one obtains in particular that $\mathbb{Q}[\mu = 1] = \mathbb{P}[\mu = 1] = p$. A standard computation based on Bayes' rule (Karatzas and Shreve 1991, Lemma 3.5.3) implies that the process $(B, \mathcal{F}^{\mu, W})$ defined by:

$$(8) \quad B_t = W_t + \left(\frac{\mu}{\sigma}\right)t; \quad t \geq 0$$

is a Brownian motion under \mathbb{Q} . Since μ is $\mathcal{F}_0^{\mu, W}$ -measurable and independent of W , and B has independent increments, B and μ are independent under \mathbb{Q} . Note that B is also a Brownian motion with respect to its own filtration \mathcal{F}^B under \mathbb{Q} . Since $dV_t = \sigma dB_t$, the filtrations \mathcal{F}^B and \mathcal{F}^V generated by B and V are identical, as well as the sets of stopping times \mathcal{T}^B and \mathcal{T}^V . We denote by $\mathbb{E}_{\mathbb{Q}}$ the expectation operator associated to \mathbb{Q} .

3.3. Properties of the value function. Define:

$$(9) \quad H_t(\mu) = \exp\left(\left(\frac{\mu}{\sigma}\right)B_t - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 t\right); \quad t \geq 0,$$

where B_t is given by (8). It follows from our previous results that $(H(\mu), \mathcal{F}^{\mu, W})$ is a martingale under \mathbb{Q} . One has the following useful lemma.

LEMMA 2. For any $\tau \in \mathcal{T}^V$,

$$\mathbb{E}[e^{-r\tau}(V_\tau - I)] = \mathbb{E}_{\mathbb{Q}}[e^{-r\tau}H_\tau(\mu)(v + \sigma B_\tau - I)].$$

Then, the following holds.

PROPOSITION 2. For any $(v, p) \in \mathbb{R} \times [0, 1]$,

$$(10) \quad G^*(v, p) = \sup_{\tau \in \mathcal{T}^B} \mathbb{E}_{\mathbb{Q}} \left[e^{-r\tau} \left\{ 1 + p \left[\exp\left(\frac{B_\tau}{\sigma} - \frac{\tau}{2\sigma^2}\right) - 1 \right] \right\} (v + \sigma B_\tau - I) \right].$$

PROOF. For any $(v, p) \in \mathbb{R} \times [0, 1]$,

$$\begin{aligned} G^*(v, p) &= \sup_{\tau \in \mathcal{T}^V} \mathbb{E}_{\mathbb{Q}}[e^{-r\tau}H_\tau(\mu)(v + \sigma B_\tau - I)] \\ &= \sup_{\tau \in \mathcal{T}^V} \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[e^{-r\tau}H_\tau(\mu)(v + \sigma B_\tau - I) \mid \mathcal{F}_\tau^V]] \\ &= \sup_{\tau \in \mathcal{T}^B} \mathbb{E}_{\mathbb{Q}}[e^{-r\tau}\mathbb{E}_{\mathbb{Q}}[H_\tau(\mu) \mid \mathcal{F}_\tau^B](v + \sigma B_\tau - I)] \\ &= \sup_{\tau \in \mathcal{T}^B} \mathbb{E}_{\mathbb{Q}}[e^{-r\tau}[pH_\tau(1) + (1-p)H_\tau(0)](v + \sigma B_\tau - I)] \\ &= \sup_{\tau \in \mathcal{T}^B} \mathbb{E}_{\mathbb{Q}} \left[e^{-r\tau} \left\{ 1 + p \left[\exp\left(\frac{B_\tau}{\sigma} - \frac{\tau}{2\sigma^2}\right) - 1 \right] \right\} (v + \sigma B_\tau - I) \right], \end{aligned}$$

where the first equality follows from Lemma 2, the third from $\mathcal{F}^V = \mathcal{F}^B$, and the fourth from the fact that, for any $\tau \in \mathcal{T}^V$, μ and B_τ are independent under \mathbb{Q} , which implies that $\mathbb{Q}[\mu = 1 \mid \mathcal{F}_\tau^B] = \mathbb{Q}[\mu = 1] = p$. \square

This result admits a natural interpretation. Under \mathbb{Q} , the value process $V^v = v + \sigma B$ is independent of μ , so that no learning occurs. We have thus transformed an incomplete information stopping problem into a complete information one. Of course, this requires a corresponding modification of the payoffs. Under \mathbb{Q} , the decision-maker maximizes the expectation of a discounted weighted average of $H(m, B)(v + \sigma B - I)$, $m = 0, 1$, where the weights $1 - p$ and p reflect his prior beliefs about μ .

The Girsanov transformation allows us to represent the belief process in terms of the value process. A direct application of Bayes' rule yields:

$$(11) \quad P_t^p = \frac{pH_t(1)}{pH_t(1) + 1 - p} = \frac{p \exp\left(\frac{V_t^v - v}{\sigma^2} - \frac{t}{2\sigma^2}\right)}{p \exp\left(\frac{V_t^v - v}{\sigma^2} - \frac{t}{2\sigma^2}\right) + 1 - p}; \quad t \geq 0,$$

see Shiriyayev (1978, §4.2.1). It is immediate from (11) that beliefs satisfy a non-crossing property, in the sense that $p_+ > p_-$ implies that $P_t^{p_+} > P_t^{p_-}$ at any time t , \mathbb{P} -almost surely.

Using the characterizations provided in Propositions 1 and 2, we can now derive some basic properties of the value function. From (5) and (6), note first that:

$$(12) \quad G^*(v, p) = \sup_{\tau \in \mathcal{T}^V} \mathbb{E} \left[e^{-r\tau} \left(v + \int_0^\tau P_t^p dt + \sigma \bar{W}_\tau - I \right) \right]; \quad (v, p) \in \mathbb{R} \times [0, 1].$$

From (12) and the non-crossing property of the belief process, we immediately obtain the following monotonicity result.

COROLLARY 1.

- (i) For any $p \in [0, 1]$, the mapping $v \mapsto G^*(v, p)$ is increasing on \mathbb{R} ;
- (ii) For any $v \in \mathbb{R}$, the mapping $p \mapsto G^*(v, p)$ is non-decreasing on $[0, 1]$.

Our next result follows immediately from (10) and the fact that the supremum of a family of linear functions is convex.

COROLLARY 2.

- (i) For any $p \in [0, 1]$, the mapping $v \mapsto G^*(v, p)$ is convex on \mathbb{R} ;
- (ii) For any $v \in \mathbb{R}$, the mapping $p \mapsto G^*(v, p)$ is convex on $[0, 1]$.

The decision-maker is thus ready to accept risky bets on the initial value v of the project, for a fixed value of p , or on the probability p that the project has a high drift, for a fixed value of v . The latter implies that costless information about μ , in the form of a mean-preserving spread over p , always has a positive value for the decision-maker. However, it does not follow from (10) that G^* is convex with respect to the pair (v, p) , and thus that the decision-maker would be ready to accept risky bets on both v and p simultaneously. Indeed, we shall argue in Section 6 that G^* cannot be convex on its whole domain.

The last result of this section concerns the continuity of the value function with respect to the initial conditions (v, p) and the variance σ of the observation-value process. In the latter case, we denote the value function by G_σ^* instead of G^* .

COROLLARY 3.

- (i) The mapping $(v, p) \mapsto G^*(v, p)$ is continuous on $\mathbb{R} \times [0, 1]$;
- (ii) The mapping $\sigma \mapsto G_\sigma^*(v, p)$ is continuous on \mathbb{R}_{++} .

4. The optimal investment region. As \mathcal{P} is intrinsically a two-dimensional problem, the standard partial differential equation approach to optimal stopping is of little help, since no closed-form solution for G^* is available. In particular, it is not clear whether the usual smooth-pasting condition holds along the investment boundary (for a discussion of this point, see Shirayayev 1978, §3.8.1). Instead of focusing on the value function, a task we shall return to in Section 5, we first determine some properties of the investment boundary function. The following result summarizes our findings.

THEOREM 1. *The investment boundary function $v^* : [0, 1] \rightarrow \mathbb{R}_{++}$ is continuous and non-decreasing on $[0, 1]$.*

Before we proceed with the proof of Theorem 1, some comments are in order. The monotonicity of the investment boundary function with respect to beliefs reflects the intuitive

idea that, for any given current value of the project, the more optimistic the decision-maker is about the drift of the value process, the more he is ready to experiment by delaying his investment. This generalizes to our incomplete information setting the standard result that the optimal investment trigger for a project whose value follows an arithmetic Brownian motion with a constant and known drift is increasing in the value of the drift.

Since $v^*(1) > v^*(0)$ and v^* is continuous on $[0, 1]$, the optimal strategy τ^* for \mathcal{P} is not a trigger strategy, that is, a stopping time of the form $\tau(\bar{v}) = \inf \{t \geq 0 \mid V_t \geq \bar{v}\}$ for some threshold \bar{v} . In contrast with predictions of standard models of irreversible investment under uncertainty (Dixit and Pindyck 1994), the value of the project at the time of investment does not therefore necessarily coincide with its maximum historic value. As pointed out in the introduction, a rational decision-maker may even optimally decide to invest after a drop of the value. The model thus allows some form of ex-post regret. For instance, the owner of a house who sells it at a discount might regret not having accepted an earlier high quote because he then anticipated a sustained boom of the housing market.

The path-dependency of the investment strategy reflects the fact that the innovations of the belief process are positively correlated with the fluctuations of the value, as is easily seen from (5). Specifically, the fluctuations of the value have two opposite effects on the investment decision. On one hand, a positive shock on V increases the payoff from investing immediately. This direct effect makes the decision-maker less willing to delay investment. On the other hand, a positive shock on V increases the subjective probability that the unknown drift μ is high, thus providing good news about the likelihood of further future increases of V . This indirect effect makes the decision-maker more willing to delay investment. These effects are reversed in the case of a negative shock on V .

Theorem 1 is established through a series of lemmas. Monotonicity and left-continuity of v^* follow from standard arguments (see for instance Villeneuve 1999, Proposition 3.2).

LEMMA 3. v^* is non-decreasing on $[0, 1]$.

PROOF. By Proposition 1, $G^*(v, p) = v - I$ for any $v \geq v^*(p)$. Moreover, the mapping $p \mapsto G^*(v, p)$ is non-decreasing according to Corollary 1. It follows that $G^*(v, p_0) = v - I$ for any $p_0 \leq p$ and $v \geq v^*(p)$, which implies that $v^*(p_0) \leq v^*(p)$. \square

LEMMA 4. v^* is left-continuous on $(0, 1]$.

PROOF. Let $\{p_n\}$ be a non-decreasing sequence in $[0, 1]$ converging to $p \in (0, 1]$. From Lemma 3, the sequence $\{v^*(p_n)\}$ is non-decreasing and bounded above by $v^*(p)$, and therefore converges to a limit $v^*(p^-)$. By definition, $G^*(v^*(p_n), p_n) = v^*(p_n) - I$ for any $n \in \mathbb{N}$. By Corollary 2, G^* is continuous, so that $G^*(v^*(p^-), p) = v^*(p^-) - I$. Hence $v^*(p) \leq v^*(p^-)$, and thus $v^*(p) = \lim_{n \rightarrow \infty} v^*(p_n)$, which implies the result. \square

The proof that v^* is right-continuous is a bit more involved. We will need the following lemma, which reflects the fact that the belief process is locally Lipschitzian with respect to its initial condition.

LEMMA 5. For any $(p_0, x) \in [0, 1] \times \mathbb{R}_{++}$, there exists $p > p_0$ such that:

$$(13) \quad \sup_{\tau \in T^V} \mathbb{E} \left[e^{-r\tau} \left[x + \int_0^\tau (P_t^p - P_t^{p_0}) dt \right] \right] = x.$$

We are now ready to complete the proof of Theorem 1.

LEMMA 6. v^* is right-continuous on $[0, 1]$.

PROOF. Let $p_0 \in [0, 1]$, and suppose that $\lim_{p \downarrow p_0} v^*(p) = v^*(p_0^+) > v^*(p_0)$. Fix some $v \in (v^*(p_0), v^*(p_0^+))$. For any $p \in (p_0, 1]$, let $\tau_{v,p}^*$ be the optimal stopping time given initial conditions (v, p) . Then, by (6) and (12),

$$\begin{aligned} G^*(v, p) &= \mathbb{E} \left[e^{-r\tau_{v,p}^*} \left(v + \int_0^{\tau_{v,p}^*} P_t^p dt + \sigma \bar{W}_{\tau_{v,p}^*} - I \right) \right] \\ (14) \quad &\leq v^*(p_0) - I + \mathbb{E} \left[e^{-r\tau_{v,p}^*} \left[v - v^*(p_0) + \int_0^{\tau_{v,p}^*} (P_t^p - P_t^{p_0}) dt \right] \right], \end{aligned}$$

where we have used the fact that $G^*(v^*(p_0), p_0) = v^*(p_0) - I$. Since $v - v^*(p_0) > 0$, Lemma 5 implies that there exists $p > p_0$ such that:

$$\mathbb{E} \left[e^{-r\tau_{v,p}^*} \left[v - v^*(p_0) + \int_0^{\tau_{v,p}^*} (P_t^p - P_t^{p_0}) dt \right] \right] \leq v - v^*(p_0).$$

Hence, from (14), $G^*(v, p) \leq v - I$ and, since the reverse inequality always hold, $G^*(v, p) = v - I$, so that $v^*(p) \leq v$ by definition of v^* . As v^* is non-decreasing by Lemma 3, it follows that $v \geq v^*(p_0^+)$. This contradicts the fact that, by assumption, $v < v^*(p_0^+)$. \square

5. Comparison with the average drift problem. In this section, we provide a comparison between the unknown drift problem and the standard problem of finding an optimal investment time for a project whose value follows an arithmetic Brownian motion with constant and known drift. Specifically, we determine whether the decision-maker prefers an investment project with unknown drift μ equal to 0 with probability $1 - p$ and to 1 with probability p , to an average drift project with a known drift equal to p .

5.1. The average drift problem. Suppose that the decision-maker has now the opportunity to invest, at cost I , into an alternative project whose value is observable and follows an arithmetic Brownian motion with known drift $p \in [0, 1]$ and known variance σ ,

$$(15) \quad d\hat{V}_t = p dt + \sigma dW_t; \quad t \geq 0.$$

Assume that the two investment projects with values V and \hat{V} are mutually exclusive and that the decision-maker must choose at date 0 the project he might later invest in. If he chooses the project with value \hat{V} , we can state his decision problem as of finding a value function $\hat{G}(\cdot, p) : \mathbb{R} \rightarrow \mathbb{R}$ and a stopping time $\hat{\tau} \in \mathcal{T}^W$ such that:

$$(16) \quad \hat{G}(v, p) = \sup_{\tau \in \mathcal{T}^W} \mathbb{E} \left[e^{-r\tau} (\hat{V}_\tau^v - I) \right] = \mathbb{E} \left[e^{-r\hat{\tau}} (\hat{V}_{\hat{\tau}}^v - I) \right]; \quad v \in \mathbb{R},$$

where v refers to the initial value of the project. We shall denote this average drift problem by $\hat{\mathcal{P}}$, in contrast with the unknown drift problem \mathcal{P} . The optimal strategy for $\hat{\mathcal{P}}$ is a trigger strategy, that is, a stopping time of the form $\hat{\tau}(\bar{v}) = \inf \{t \geq 0 \mid \hat{V}_t^v \geq \bar{v}\}$ for some threshold \bar{v} . We have the following result.

LEMMA 7. For any $p \in [0, 1]$, let $f(p) = \sqrt{p^2 + 2r\sigma^2} - p$ and $\hat{v}(p) = I + \sigma^2/f(p)$. Then $\hat{\tau}(\hat{v}(p))$ is an optimal stopping time for $\hat{\mathcal{P}}$.

Lemma 7 is standard, and results from the fact that the Laplace transform of $\hat{\tau}(\bar{v})$ is given by $\mathbb{E}[e^{-r\hat{\tau}(\bar{v})}] = \exp((v - \bar{v})f(p)/\sigma^2)$ (Karatzas and Shreve 1991, §3.5.C). The value function for $\hat{\mathcal{P}}$ can then be written as:

$$(17) \quad \hat{G}(v, p) = \begin{cases} \exp\left(-1 + \frac{f(p)}{\sigma^2}(v - I)\right) \frac{\sigma^2}{f(p)} & \text{if } v < I + \frac{\sigma^2}{f(p)}, \\ v - I & \text{if } v \geq I + \frac{\sigma^2}{f(p)}. \end{cases}$$

There is no a priori obvious relationship between G^* and \hat{G} , except of course when p is equal to 0 or 1, in which case the decision-maker in \mathcal{P} knows with certainty the value of the drift, so that $G^*(\cdot, 0) = \hat{G}(\cdot, 0)$ and $G^*(\cdot, 1) = \hat{G}(\cdot, 1)$. With a slight abuse of terminology, we will refer to \hat{v} as the investment boundary function for the average drift problem.

5.2. The comparison result. Since the objective function in (16) is linear in the drift p of \hat{V} , the value function \hat{G} for the average drift problem $\hat{\mathcal{P}}$ is convex in p . This implies that a risk-neutral decision-maker is ready to exchange the option to invest in the average drift problem for an option to invest in a project with value $V_t^v = v + \mu t + \sigma W_t$ and uncertain drift $\mu \in \{0, 1\}$ drawn according to $\mathbb{P}[\mu = 1] = p$, provided he is informed of the value of μ immediately after its realization, and can thus take his investment decision under complete information about μ . In the unknown drift problem \mathcal{P} , the information structure of the decision-maker is coarser, since he has only access to an imperfect learning technology, namely, the observation of V . Nevertheless, we have the following result.

THEOREM 2. *For any $(v, p) \in \mathbb{R} \times [0, 1]$,*

$$G^*(v, p) \geq \hat{G}(v, p).$$

Thus the value of the unknown drift problem is higher than that of the average drift problem, despite the fact that, in the former, the decision-maker has to acquire information about the drift before taking his investment decision. Geometrically, in the value-belief space, the investment boundary v^* for the unknown drift problem \mathcal{P} is to the right of the investment boundary \hat{v} for the average drift problem $\hat{\mathcal{P}}$, that is $v^* \geq \hat{v}$, see Figure 1 below.

The proof of Theorem 2 relies on standard discrete-time approximations of \mathcal{P} and $\hat{\mathcal{P}}$. Let Ψ be the space of continuous $\psi : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ such that the family $\{\psi(V_\tau^v, P_\tau^p) \mid \tau \in \mathcal{T}^V\}$ is uniformly integrable, and let $\hat{\Psi}$ be the space of continuous $\hat{\psi} : \mathbb{R} \rightarrow \mathbb{R}$ such that the family $\{\hat{\psi}(\hat{V}_\tau^v) \mid \tau \in \mathcal{T}^W\}$ is uniformly integrable. For any fixed $\delta > 0$, interpreted as the duration of a discrete-time period, define operators acting respectively on Ψ and $\hat{\Psi}$ by:

$$(18) \quad Q_\delta(\psi)(v, p) = \max\left\{\psi(v, p), \mathbb{E}\left[e^{-r\delta}\psi(V_\delta^v, P_\delta^p)\right]\right\}; \quad \psi \in \Psi, (v, p) \in \mathbb{R} \times [0, 1],$$

and:

$$(19) \quad \hat{Q}_\delta(\hat{\psi})(v) = \max\left\{\hat{\psi}(v), \mathbb{E}\left[e^{-r\delta}\hat{\psi}(\hat{V}_\delta^v)\right]\right\}; \quad \hat{\psi} \in \hat{\Psi}, v \in \mathbb{R}.$$

It is clear from these definitions that Q_δ and \hat{Q}_δ map respectively Ψ and $\hat{\Psi}$ into themselves. Therefore, for any $\psi \in \Psi$, $\hat{\psi} \in \hat{\Psi}$, and $n \in \mathbb{N}$, one can define $Q_\delta^n(\psi)$ and $\hat{Q}_\delta^n(\hat{\psi})$ recursively. Two key properties are that Q_δ is monotone, that is $Q_\delta(\phi) \leq Q_\delta(\psi)$ whenever $\phi \leq \psi$, and that \hat{Q}_δ preserves convexity by the convexity of the maximum operator and of the process \hat{V}^v with respect to its initial condition v .

For any $v \in \mathbb{R}$, let $\hat{\psi}(v) = v - I$, and let $\pi_1 : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ be the projection of elements of $\mathbb{R} \times [0, 1]$ on their first coordinate. It is immediate to check that $\hat{\psi} \in \hat{\Psi}$ and $\hat{\psi} \circ \pi_1 \in \Psi$. Our next result follows immediately from the characterization of the value function of an optimal stopping problem as the smallest excessive upper bound of the reward function (Shiryayev 1978, §3.2.2, Lemma 3, and §3.3.1, Theorem 1).

LEMMA 8. *For any $(v, p) \in \mathbb{R} \times [0, 1]$,*

- (i) $\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} Q_\delta^n(\hat{\psi} \circ \pi_1)(v, p) = G^*(v, p);$
- (ii) $\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \hat{Q}_\delta^n(\hat{\psi})(v) = \hat{G}(v, p).$

The interpretation of this approximation result is clear. For any fixed $\delta > 0$, the limit with respect to n yields the value of the infinite-horizon, discrete-time problem, where the investor is constrained to stopping times with range in $\{n\delta \mid n \in \mathbb{N}\}$. Letting then δ go to 0 yields the value of the continuous-time problem. Given Lemma 8, Theorem 2 is a direct consequence of the following lemma.

LEMMA 9. *For any $(\delta, n) \in \mathbb{R}_{++} \times \mathbb{N}$,*

$$Q_\delta^n(\hat{\psi} \circ \pi_1) \geq \hat{Q}_\delta^n(\hat{\psi}) \circ \pi_1.$$

PROOF. We proceed by induction. First, for any $(v, p) \in \mathbb{R} \times [0, 1]$,

$$\begin{aligned} Q_\delta(\hat{\psi} \circ \pi_1)(v, p) &= \max\left\{\hat{\psi}(v), \mathbb{E}\left[e^{-r\delta}\hat{\psi}(v + \mu\delta + \sigma W_\delta)\right]\right\} \\ &\geq \max\left\{\hat{\psi}(v), \mathbb{E}\left[e^{-r\delta}\hat{\psi}(v + p\delta + \sigma W_\delta)\right]\right\} \\ &= (\hat{Q}_\delta(\hat{\psi}) \circ \pi_1)(v, p), \end{aligned}$$

where the inequality follows from the convexity of $\hat{\psi}$ and the independence between μ and W , together with Jensen's inequality. Next, suppose that $Q_\delta^n(\hat{\psi} \circ \pi_1) \geq \hat{Q}_\delta^n(\hat{\psi}) \circ \pi_1$ for some $n \in \mathbb{N}$. Then, for any $(v, p) \in \mathbb{R} \times [0, 1]$,

$$\begin{aligned} Q_\delta^{n+1}(\hat{\psi} \circ \pi_1)(v, p) &= Q_\delta(Q_\delta^n(\hat{\psi} \circ \pi_1))(v, p) \\ &\geq Q_\delta(\hat{Q}_\delta^n(\hat{\psi}) \circ \pi_1)(v, p) \\ &\geq (\hat{Q}_\delta(\hat{Q}_\delta^n(\hat{\psi})) \circ \pi_1)(v, p) \\ &= (\hat{Q}_\delta^{n+1}(\hat{\psi}) \circ \pi_1)(v, p), \end{aligned}$$

where the first inequality follows from the induction hypothesis and the monotonicity of Q_δ , and the second inequality from the first part of the proof together with the fact that \hat{Q}_δ preserves convexity. Hence the result. \square

5.3. A remark on trigger strategies. In the previous subsection, we have used dynamic programming techniques to compare the unknown drift problem \mathcal{P} with the average drift problem $\hat{\mathcal{P}}$. Since the optimal stopping time in $\hat{\mathcal{P}}$ is a trigger strategy, one might wonder

whether a more direct approach would not consist to restrict the strategy space in \mathcal{P} to trigger strategies conditional on the current value V of the project, and to compare the value of this constrained problem to that of $\hat{\mathcal{P}}$. While this approach turns out not to be conclusive, it is nevertheless instructive to understand why. To do so, let $\bar{\mathcal{T}}^V$ be the subset of \mathcal{T}^V composed of stopping times of the form $\tau(\bar{v}) = \inf\{t \geq 0 \mid V_t \geq \bar{v}\}$ for some threshold \bar{v} , and consider the following constrained problem:

$$(20) \quad \bar{G}(v, p) = \sup_{\tau \in \bar{\mathcal{T}}^V} \mathbb{E}[e^{-r\tau} (V_\tau^v - I)].$$

Using (10), a standard computation based on Girsanov's theorem (Karatzas and Shreve 1991, Corollary 3.5.2) and on the formula for the Laplace transform of the passage time to a level \bar{v} for an arithmetic Brownian motion yields the following result.

LEMMA 10. *For any $(v, p) \in \mathbb{R} \times [0, 1]$,*

$$(21) \quad \bar{G}(v, p) = \max_{\bar{v} \geq v} \Gamma(v, p, \bar{v}),$$

where, for any $\bar{v} \in \mathbb{R}$,

$$(22) \quad \Gamma(v, p, \bar{v}) = \left[p \exp\left(\frac{v - \bar{v}}{\sigma^2} f(1)\right) + (1 - p) \exp\left(\frac{v - \bar{v}}{\sigma^2} f(0)\right) \right] (\bar{v} - I).$$

The expression (22) for the objective in (21) admits a natural interpretation. When choosing an optimal trigger \bar{v} in \mathcal{P} , the decision-maker maximizes a weighted average of the payoffs he would get from playing $\tau(\bar{v})$ if he knew the true value of the drift, where the weights $1 - p$ and p reflect his prior beliefs about μ . It is clear from (22) that the choice of an optimal trigger in (21) depends on the initial value v of the project as well as on his prior belief p . Thus this choice is not dynamically consistent, in the sense that a decision-maker constrained to trigger strategies would like to revise his choice of a trigger strategy as the value changes and new information about the drift becomes available. Formulas (21)–(22) do not allow to derive a closed-form solution for the optimal trigger $\bar{v}(v, p)$ conditional on the initial state (v, p) . However, one can unambiguously compare $\bar{v}(v, p)$ with the optimal trigger $\hat{v}(p)$ for the average drift problem, at least for specific values of v and p . Specifically, let $\tilde{v}(p) = I + \sigma^2/[pf(1) + (1 - p)f(0)]$ for any $p \in [0, 1]$. Then one has the following result.

PROPOSITION 3. *For any $p \in (0, 1)$ and $v \geq \tilde{v}(p)$, $\bar{v}(v, p) = v$.*

Since f is convex and positive, $\tilde{v}(p) < \hat{v}(p)$ whenever $p \in (0, 1)$. Hence, for any such p and $v \in [\tilde{v}(p), \hat{v}(p))$, $\bar{v}(v, p) < \hat{v}(p)$ and thus $\bar{G}(v, p) = v - I < \hat{G}(v, p)$. For these values of (v, p) , the decision-maker, when constrained to use trigger strategies, invests as if the drift were constant and equal to $f^{-1}(\sigma^2/(v - I)) < p$, making him just ready to invest immediately. This reflects an implicit risk-aversion due to the additional uncertainty generated by the randomness of μ . Clearly, Proposition 3 does not allow to compare \mathcal{P} and $\hat{\mathcal{P}}$. In a sense, this is not surprising, as the restriction to trigger strategies in \mathcal{P} essentially amounts to deprive the decision-maker from the benefits of learning about the unknown drift.

6. A local study of the investment boundary. While the comparison result of Theorem 2 provides a useful lower bound for the value function G^* of problem \mathcal{P} , it says little about the shape of the investment boundary function v^* , nor about the wedge between the unknown drift problem and the average drift problem. We now address these and related questions in the case in which the variance σ of the observation-value process is small.

6.1. A strict comparison result. In the remaining of the paper, we systematically index the value processes V and \hat{V} , the value functions G^* and \hat{G} , and the investment boundary functions v^* and \hat{v} by the variance parameter σ . Our objective in this section is to compare G_σ^* and \hat{G}_σ , as well as the investment boundaries v_σ^* and \hat{v}_σ for small values of σ . Specifically, consider the open domain $\mathcal{D} = (I, I + 1/r) \times (0, 1)$. Our discussion will be based on the following strict comparison result.

THEOREM 3. *For any compact set $K \subset \mathcal{D}$, there exists $\sigma_K > 0$ such that for any $(\sigma, v, p) \in (0, \sigma_K] \times K$,*

$$G_\sigma^*(v, p) > \hat{G}_\sigma(v, p).$$

The proof of Theorem 3 is based on three uniform convergence lemmas. First, we study the solution to the average drift problem $\tilde{\mathcal{P}}$ as σ goes to 0. Next, we introduce an auxiliary problem $\tilde{\mathcal{P}}$ which yields a value uniformly close to that of \mathcal{P} as σ goes to 0. We conclude by proving that $\tilde{\mathcal{P}}$ yields a strictly higher value than $\hat{\mathcal{P}}$ as σ goes to 0.

First, we study the behavior of \hat{G}_σ when the variance σ of \hat{V}_σ becomes arbitrarily small. When $\sigma = 0$, solving $\hat{\mathcal{P}}$ simply amounts to finding a maximum of $t \mapsto e^{-rt}(v + pt - I)$, yielding $\hat{t}_0(v, p) = \max\{0, (I - v)/p + 1/r\}$ whenever $p > 0$ and $\hat{t}_0(v, 0) = 0$ whenever $v > I$. It is then straightforward to check that, for any $(v, p) \in \mathcal{D}$,

$$(23) \quad \hat{G}_0(v, p) = \begin{cases} \exp\left(-1 + \frac{r}{p}(v - I)\right) \frac{p}{r} & \text{if } v < I + \frac{p}{r}, \\ v - I & \text{if } v \geq I + \frac{p}{r}. \end{cases}$$

A key observation is that, for any initial value v , $\hat{G}_0(v, p)$ is convex in p . In particular,

$$(24) \quad p\hat{G}_0(v, 1) + (1 - p)\hat{G}_0(v, 0) > \hat{G}_0(v, p); \quad (v, p) \in \mathcal{D}.$$

One has the following result.

LEMMA 11. $\lim_{\sigma \rightarrow 0} \hat{G}_\sigma(v, p) = \hat{G}_0(v, p)$ uniformly in (v, p) on any compact set $K \subset \mathcal{D}$.

To compare G_σ^* and \hat{G}_σ , it will be helpful to consider an auxiliary optimal stopping problem that differs from \mathcal{P} only in that the Gaussian component of the value is omitted in the decision-maker's payoff:

$$(25) \quad \tilde{G}_\sigma(v, p) = \sup_{\tau \in \mathcal{T}^{\mathcal{V}_\sigma}} \mathbb{E}[e^{-r\tau}(v + \mu\tau - I)].$$

We shall denote this problem by $\tilde{\mathcal{P}}$. Note that the information structure is the same in \mathcal{P} and $\tilde{\mathcal{P}}$. This leads to the following result.

LEMMA 12. $\lim_{\sigma \rightarrow 0} G_\sigma^*(v, p) - \tilde{G}_\sigma(v, p) = 0$ uniformly in (v, p) on $\mathbb{R} \times [0, 1]$.

It should be noted that the decision-maker can secure the payoff $\hat{G}_0(v, p)$ in \mathcal{P} or $\tilde{\mathcal{P}}$ by mimicking the optimal strategy in $\tilde{\mathcal{P}}$ when $\sigma = 0$, that is, by delaying investment by the deterministic time amount $\hat{t}_0(v, p)$. This follows immediately from the linearity of the payoff functions in (6) and (25) with respect to μ . We now prove that, as σ goes to 0, \tilde{G}_σ converges uniformly to the function $v \mapsto p\hat{G}_0(v, 1) + (1 - p)\hat{G}_0(v, 0)$, which implies Theorem 3 given Lemmas 11 and 12 and inequality (24).

LEMMA 13. $\lim_{\sigma \rightarrow 0} \tilde{G}_\sigma(v, p) = p\hat{G}_0(v, 1) + (1 - p)\hat{G}_0(v, 0)$ uniformly in (v, p) on any compact set $K \subset \mathcal{D}$.

The intuition is as follows. As already mentioned, the decision-maker can always secure the payoff $\hat{G}_0(v, p)$ in $\hat{\mathcal{P}}$. In fact, he can do strictly better by first waiting a deterministic time amount $\varepsilon > 0$ and only then playing the optimal strategy in $\hat{\mathcal{P}}$ for $\sigma = 0$, conditional on the expected value of $(v + \mu\varepsilon, \mu)$ at time ε . The delay ε corresponds to a learning phase, during which the decision-maker accumulates information about μ before taking his decision. As σ goes to 0, the accuracy of V_σ as a signal of μ becomes infinite, so this learning phase can be made arbitrarily short. In the limit, everything happens as if the decision-maker knew exactly the value of μ at date 0, which implies pointwise convergence.

The fact that convergence is uniform in (v, p) on any compact subset of \mathcal{D} follows from a time-change argument. For a fixed duration ε of the learning phase, reducing the variance of the observation process effectively amounts to increasing the duration of the learning phase while keeping the variance constant. Since beliefs follow a martingale, this generates a mean-preserving spread in the beliefs of the decision-maker. As his expected gain at the end of the learning phase is convex in $(v + \varepsilon P_\varepsilon, P_\varepsilon)$, the payoff from this investment strategy increases as σ decreases to 0, which implies the result.

6.2. Comments and interpretations. An immediate consequence of Lemmas 11, 12, and 13 is that $\lim_{\sigma \rightarrow 0} G_\sigma^*(v, p) - [p\hat{G}_\sigma(v, 1) + (1-p)\hat{G}_\sigma(v, 0)] = 0$ uniformly on any compact subset of \mathcal{D} . This means that, as σ goes to 0, the value of the unknown drift problem \mathcal{P} converges uniformly to the value of an investment problem in which the drift μ of the value process is first selected according to a lottery on 0 and 1 with respective probabilities $1-p$ and p , and immediately revealed to the decision-maker, who then takes his investment decision under complete information about μ . The loss in value arising from the need to learn about the drift vanishes as the variance of the observation process converges to 0. The fact that the unknown drift problem \mathcal{P} is strictly preferred by the decision-maker to the average drift problem $\hat{\mathcal{P}}$ for small values of σ simply reflects the fact that the value function \hat{G}_σ of $\hat{\mathcal{P}}$ is convex on \mathcal{D} with respect to p , and that learning about μ in \mathcal{P} is fast when σ is small.

From Lemma 7, the optimal investment strategy in the average drift problem $\hat{\mathcal{P}}$ with drift p consists to delay investment until the value \hat{V} hits the threshold $\hat{v}_\sigma(p)$. It is easy to check that, as σ goes to 0, \hat{v}_σ converges monotonically from above to the mapping $p \mapsto I + p/r$. This implies that any pair $(v, p) \in \mathcal{D}' = \{(v, p) \in \mathcal{D} \mid v > I + p/r\}$ satisfies $\hat{G}_\sigma(v, p) = v - I$ for σ close enough to 0. It follows then from Theorem 3 that $G_\sigma^*(v, p) > v - I$, that is, (v, p) belongs to the continuation region of \mathcal{P} for σ close enough to 0. Moreover, this reasoning can be made uniform on compact subsets of \mathcal{D} .

COROLLARY 4. *For any compact subset $K \subset \mathcal{D}'$, there exists $\sigma_K > 0$ such that for any $(\sigma, v, p) \in (0, \sigma_K] \times K$,*

$$G_\sigma^*(v, p) > v - I = \hat{G}_\sigma(v, p).$$

We can actually characterize exactly the asymptotic behavior of the investment boundary function v_σ^* as σ converges to 0.

COROLLARY 5.

- (i) $\lim_{\sigma \rightarrow 0} v_\sigma^*(0) = I$;
- (ii) For any $p \in (0, 1]$, $\lim_{\sigma \rightarrow 0} v_\sigma^*(p) = I + 1/r$.

The optimal investment boundary function v_σ^* converges pointwise to the discontinuous function $v_0^* = I + 1_{(0,1]}/r$ as σ goes to 0. The proof simply consists to apply Corollary 4 to an increasing sequence $\{K_n\}$ of compact subsets of \mathcal{D}' such that $\bigcup_{n=0}^\infty K_n = \mathcal{D}'$. To interpret this

result, consider the limit problem arising from \mathcal{P} when the Gaussian component is omitted both in the information structure of the decision-maker and in his payoff:

$$(26) \quad G_0^*(v, p) = \sup_{\tau \in \mathcal{T}^{V_0}} \mathbb{E}[e^{-r\tau}(v + \mu\tau - I)].$$

As the signal V_0 is perfect, all learning about μ takes place at date 0, and the belief process jumps instantaneously to one of its absorbing barriers, 0 with probability $1 - p$, or 1 with probability p . If $v \geq I$ then, in the first case, it is optimal to invest immediately, while in the second, it is optimal to wait until the value reaches the threshold $I + 1/r$. Thus v_0^* can be interpreted as the investment boundary for the limit problem (26). It should be noted that $G_0^*(v, p) = p\hat{G}_0(v, 1) + (1 - p)\hat{G}_0(v, 0)$, in line with Lemma 13. The triangular area \mathcal{D}' therefore represents the discrepancy between the unknown drift problem and the average drift problem as σ goes to 0. Our results are illustrated on Figure 1.

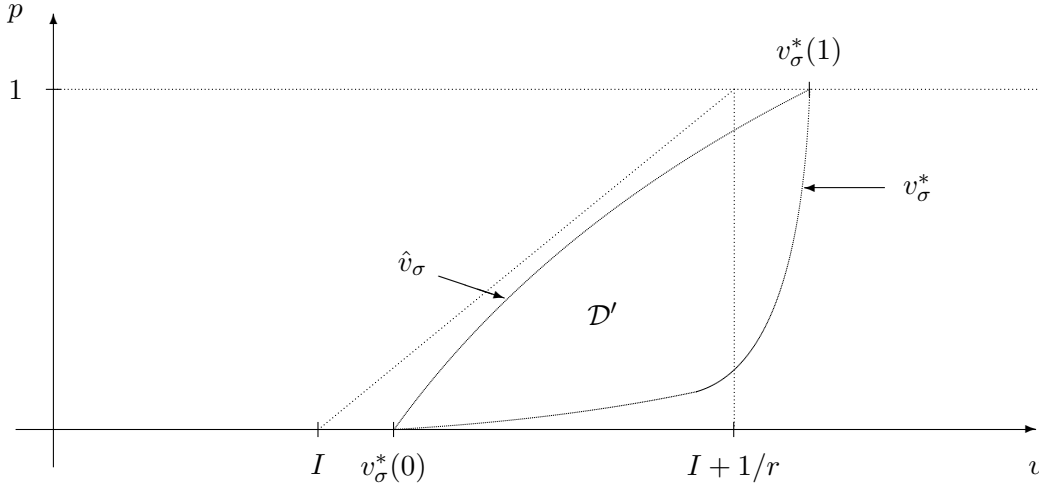


FIGURE 1. Local comparison between \mathcal{P} and $\hat{\mathcal{P}}$.

While \hat{v}_σ is unambiguously convex, the concave shape of v_σ^* is only meant to be suggestive. However, in virtue of Corollary 5, v_σ^* cannot be globally convex on $[0, 1]$ when σ is small. This implies in particular that the value function G_σ^* is not globally convex with respect to the value-belief pair. Indeed, if it were, then, for any two points (v, p) and (\tilde{v}, \tilde{p}) on the investment boundary, and for any convex combination (v_λ, p_λ) of these points, we would have $G_\sigma^*(v_\lambda, p_\lambda) \leq \lambda G_\sigma^*(v, p) + (1 - \lambda)G_\sigma^*(\tilde{v}, \tilde{p}) = v_\lambda - I$, so that $G_\sigma^*(v_\lambda, p_\lambda) = v_\lambda - I$ and (v_λ, p_λ) would belong to the investment region as well. But this can only hold if v_σ^* is globally convex on $[0, 1]$, a contradiction.

The convergence of the investment boundary v_σ^* to the discontinuous function v_0^* as σ goes to 0 has a striking consequence. Indeed, consider some $(v, p) \in \mathcal{D}'$ such that, for some $\sigma > 0$, $G_\sigma^*(v, p) = v - I$, and thus it is optimal to invest in the state (v, p) when the observation-value process has variance σ . It is clearly possible to find a triple (v, p, σ) satisfying this condition, since v_σ^* is continuous with respect to p and $v_\sigma^*(0)$ is close to I for small enough σ . But since $v < I + 1/r$ and $p > 0$, Corollary 5 implies that, for all $\tilde{\sigma} < \sigma$ that are close enough to 0, $v_{\tilde{\sigma}}^*(p) > v$ and thus $G_{\tilde{\sigma}}^*(v, p) > v - I$. In other terms, the value of problem \mathcal{P} can be a decreasing function of the variance σ of the value process, at least locally. Our findings are illustrated on Figure 2. Here, $\tilde{\sigma} < \sigma$, and, on the hatched zone, $G_{\tilde{\sigma}}^*(v, p) > G_\sigma^*(v, p) = v - I$. Again, the exact shapes of v_σ^* and $v_{\tilde{\sigma}}^*$, as well as the fact that they cross only twice, are only meant to be suggestive.

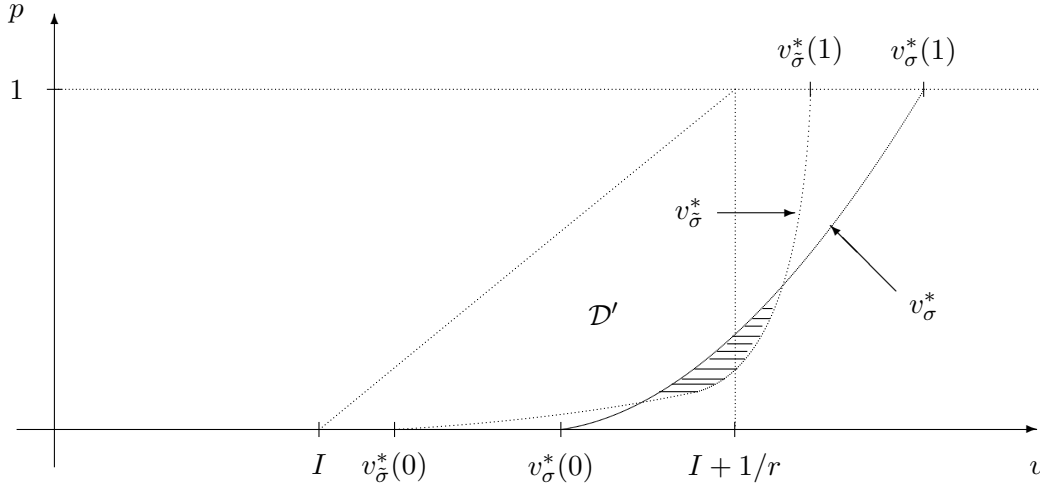


FIGURE 2. Non-monotonicity with respect to σ .

This non-monotonicity result contrasts sharply with the predictions of standard real option models (Dixit and Pindyck 1994, §5.4). There, a greater uncertainty (in the sense of a higher σ) typically increases the value of a firm's investment opportunity, and increases the critical value at which investment takes place by raising the opportunity cost of exercising the option to invest. A similar result also holds for the average drift problem, as \hat{v}_σ and therefore \hat{G}_σ are clearly increasing in σ . The intuition is that when the variance increases, the decision-maker can achieve a higher exposition to upside realizations of the value by increasing his investment trigger, while being protected from downside risk.

By contrast, in the incomplete information problem, the decision-maker is not protected from downside risk, since the investment boundary is not flat in p . An increase in σ has thus an ambiguous effect on the value of the option to invest, because of the interplay between two opposite effects. On the one hand, an increase in σ raises the variance of the payoff of the decision-maker at the time of investment, which tends to increase the value of the option to invest. One might call this standard effect the *real option effect* by analogy with the complete information case. On the other hand, a raise in σ decreases the variance of the belief process, which tends to impede learning about μ , and thus to reduce the opportunity cost to exert the option to invest. One might call this countervailing effect the *inefficient learning effect*. On the hatched zone in Figure 2, the inefficient learning effect clearly dominates. In this zone, a reduction in σ is likely to delay investment. This casts some doubt on the effectiveness of policies aiming at promoting investment by reducing the level of uncertainty, at least when such a reduction in uncertainty facilitates learning.

Overall, the impact of an increase in σ on the value of the option to invest depends on which of the real option and the inefficient learning effects dominates. It is difficult to map precisely the parameter space in terms of this distinction. Note however that, since $v_\sigma^*(0)$ and $v_\sigma^*(1)$ are both increasing functions of σ and $v_\sigma^*(p)$ is a continuous non-decreasing function of p , $v_\sigma^*(p)$ must at least be locally increasing in σ in neighborhoods of 0 and 1, as well as $G_\sigma^*(v, p)$ for v close enough to $v_\sigma^*(p)$ (see Figure 2 for an illustration of this effect). Intuitively, if the decision-maker is already fairly confident in his estimate of μ , an increase in σ will only have a marginal impact on the efficiency of learning, since his beliefs are unlikely to change very fast anyway. The increased variance of his payoff make him however willing to delay his investment further, thereby increasing the value of his option to invest. In that case, the real option effect compensates for the decreased efficiency of learning.

7. Extensions. In this section, we discuss some extensions of our model. First, we allow the unknown drift of the value process to take an arbitrary finite number of possibly negative values. Next, we discuss the problems arising when the drift is allowed to take an infinite number of values. Last, we study the robustness of our results to a geometric Brownian motion specification of the value process.

7.1. Finitely many drift values. Our basic assumptions are unchanged, except that μ can now take any of $M \geq 2$ values $m^1 \leq \dots \leq m^M$ in \mathbb{R} . For any $i = 1, \dots, M$, we denote by $p^i = \mathbb{P}[\mu = m^i]$ the prior belief of the decision-maker that $\mu = m^i$, and we let $p = (p^1, \dots, p^M)$. At any time t , the belief of the decision-maker about μ is represented by a vector $P_t = (P_t^1, \dots, P_t^M)$, where $P_t^i = \mathbb{P}[\mu = m^i \mid \mathcal{F}_t^V]$ for any $i = 1, \dots, M$. From Liptser and Shiryaev (1977, Theorems 7.12 and 9.1), the belief process (P, \mathcal{F}^V) satisfies the following filtering equation:

$$(27) \quad dP_t^i = \frac{P_t^i \left(m^i - \sum_{j=1}^M P_t^j m^j \right)}{\sigma} d\bar{W}_t; \quad t \geq 0, i = 1, \dots, M,$$

where the innovation process (\bar{W}, \mathcal{F}^V) is a standard Wiener process given by:

$$(28) \quad d\bar{W}_t = \frac{dV_t - \sum_{j=1}^M P_t^j m^j dt}{\sigma}; \quad t \geq 0.$$

The recursive formulation of the problem is again given by (6), bearing in mind that (P^p, \mathcal{F}^V) is now a vector-valued process with values in the unit simplex Δ of \mathbb{R}^M . The following result parallels Proposition 1.

PROPOSITION 4.

- (i) *There exist a value function G^* and a stopping time $\tau^* \in \mathcal{T}^V$ that solve (6);*
- (ii) *The coincidence set $\mathcal{S}^* = \{(v, p) \in \mathbb{R} \times \Delta \mid G^*(v, p) = g(v, p)\}$ for (6) is non-empty and satisfies $\mathcal{S}^* = \bigcup_{p \in \Delta} \{p\} \times [v^*(p), \infty)$, where $v^*(p) = \inf\{v \in \mathbb{R} \mid (v, p) \in \mathcal{S}^*\} < \infty$ for any $p \in \Delta$.*

A key difference with Proposition 1 is that the optimal stopping time is no longer \mathbb{P} -almost surely finite if the drift can take negative values. The investment boundary function v^* now determines a surface in $\mathbb{R} \times \Delta$, and it is optimal to invest whenever the value-belief process crosses this surface. The characterization of the value function G^* is analogous to that provided in Section 3. Specifically, the Girsanov transformation extends straightforwardly to the multiple drift case, as well as Lemma 2, which leads to the following corollary of Proposition 2.

COROLLARY 6. *For any $(v, p) \in \mathbb{R} \times \Delta$,*

$$(29) \quad G^*(v, p) = \sup_{\tau \in \mathcal{T}^B} \mathbb{E}_{\mathbb{Q}} \left[e^{-r\tau} \sum_{i=1}^M p^i \exp \left(\left(\frac{m^i}{\sigma} \right) B_\tau - \frac{1}{2} \left(\frac{m^i}{\sigma} \right)^2 \tau \right) (v + \sigma B_\tau - I) \right].$$

As in (11), the belief process can be represented in terms of the value process:

$$(30) \quad P_t^{i,p} = \frac{p^i H_t(m^i)}{\sum_{j=1}^M p^j H_t(m^j)} = \frac{p^i \exp \left(\left(\frac{m^i}{\sigma} \right) \frac{V_t^v - v}{\sigma} - \frac{1}{2} \left(\frac{m^i}{\sigma} \right)^2 t \right)}{\sum_{j=1}^M p^j \exp \left(\left(\frac{m^j}{\sigma} \right) \frac{V_t^v - v}{\sigma} - \frac{1}{2} \left(\frac{m^j}{\sigma} \right)^2 t \right)}; \quad t \geq 0,$$

and the analogue of (12) is:

$$(31) \quad G^*(v, p) = \sup_{\tau \in T^V} \mathbb{E} \left[e^{-r\tau} \left(v + \int_0^\tau \sum_{i=1}^M P_t^{i,p} m^i dt + \sigma \bar{W}_\tau - I \right) \right]; \quad (v, p) \in \mathbb{R} \times \Delta.$$

From (29) and (31), it is clear that Corollary 1(i) and Corollary 2 still hold, as well as the continuity properties of Corollary 3. The multi-dimensional analogue of Corollary 1(ii) can be stated as follows.

COROLLARY 7. *Suppose that p and \tilde{p} are ordered in the sense of the monotone likelihood ratio property, that is, p^i/\tilde{p}^i is increasing in $i = 1, \dots, M$. Then $G^*(v, p) \geq G^*(v, \tilde{p})$.*

The proof is an immediate consequence of (31) together with the fact that Bayesian updating preserves the monotone likelihood property. The comparison result with the average drift problem is unaffected. To see this, redefine the dynamics of \hat{V} as:

$$(32) \quad d\hat{V}_t = \sum_{i=1}^M p^i m^i dt + \sigma dW_t; \quad t \geq 0.$$

Lemmas 8 and 9 remain true, provided p is replaced by the prior expectation of the drift $\sum_{i=1}^M p^i m^i$. Thus Theorem 2 holds for an arbitrary finite number of values for the drift. Our approach of the strict comparison result can be extended provided the largest possible value for the drift, m^M , is strictly positive. One needs only to redefine the domain \mathcal{D} as $(I, I + m^M/r) \times \Delta$, and the key inequality (24) is replaced by:

$$(33) \quad \sum_{i=1}^M p^i \hat{G}_0(v, m^i) > \hat{G}_0 \left(v, \sum_{i=1}^M p^i m^i \right); \quad (v, p) \in \mathcal{D}.$$

The analogues of Lemmas 11, 12 and 13 are easily derived, from which Theorem 3 follows. The asymptotic behavior of the investment boundary function v_σ^* as σ converges to 0 can then be straightforwardly characterized along the lines of Subsection 6.2. For instance, if the drift can take three values $-1, 0$ and 1 , one has $\lim_{\sigma \rightarrow 0} v_\sigma^*(1, 0, 0) = \lim_{\sigma \rightarrow 0} v_\sigma^*(0, 1, 0) = I$ and $\lim_{\sigma \rightarrow 0} v_\sigma^*(p) = I + 1/r$ for any $p \in \Delta \setminus \{(1, 0, 0), (0, 1, 0)\}$.

7.2. Infinitely many drift values. The situation is significantly more complex when μ can take an infinite number of values. Let \mathbb{M} be the probability measure on the Borel σ -field $\mathcal{B}(\mathbb{R})$ describing the probability law of μ , and suppose that $\mu \in L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{M})$. Proceeding as in Subsection 3.2, one can construct a probability measure \mathbb{Q} under which V and μ are independent, and whose restriction to $\mathcal{F}_t^{\mu, W}$ has a Radon–Nikodym derivative with respect to \mathbb{P} given by (7). The process $(B, \mathcal{F}^{\mu, W})$ given by (8) remains a Brownian motion under \mathbb{Q} . A direct application of Bayes' rule (Karatzas and Shreve 1991, Lemma 3.5.3) yields that:

$$\mathbb{E}[\mu \mid \mathcal{F}_t^V] = \frac{\mathbb{E}_{\mathbb{Q}} \left[\exp \left(\left(\frac{\mu}{\sigma} \right) \frac{V_t^v - v}{\sigma} - \frac{1}{2} \left(\frac{\mu}{\sigma} \right)^2 t \right) \mu \mid \mathcal{F}_t^V \right]}{\mathbb{E}_{\mathbb{Q}} \left[\exp \left(\left(\frac{\mu}{\sigma} \right) \frac{V_t^v - v}{\sigma} - \frac{1}{2} \left(\frac{\mu}{\sigma} \right)^2 t \right) \mid \mathcal{F}_t^V \right]} = \Phi(t, V_t^v); \quad t \geq 0,$$

where, by definition:

$$\Phi(t, x) = \frac{\int_{\mathbb{R}} \exp \left(\left(\frac{m}{\sigma} \right) \frac{x - v}{\sigma} - \frac{1}{2} \left(\frac{m}{\sigma} \right)^2 t \right) m \mathbb{M}(dm)}{\int_{\mathbb{R}} \exp \left(\left(\frac{m}{\sigma} \right) \frac{x - v}{\sigma} - \frac{1}{2} \left(\frac{m}{\sigma} \right)^2 t \right) \mathbb{M}(dm)}; \quad x \in \mathbb{R}.$$

One can then decompose the value process in its own filtration and obtain:

$$(34) \quad dV_t^v = \Phi(t, V_t^v)dt + \sigma d\bar{W}_t; \quad t \geq 0,$$

where (\bar{W}, \mathcal{F}^V) is a standard Wiener process. The investment problem (2) can then be rewritten as follows:

$$(35) \quad \sup_{\tau \in T^V} \mathbb{E} \left[e^{-r\tau} \left[v + \int_0^\tau \Phi(s, V_s^v) ds + \sigma \bar{W}_\tau - I \right] \right].$$

In contrast with (5)–(6), a key feature of (34)–(35) is that it does not allow in general a finite-dimensional Markov representation. This represents a challenging problem that is beyond the scope of the present paper.

7.3. Geometric Brownian motion. Returning for simplicity to the case in which μ can take only the values 0 and 1, suppose now that the value of the project follows a geometric Brownian motion with unknown drift μ and known variance σ ,

$$(36) \quad dV_t = \mu V_t dt + \sigma V_t dW_t; \quad t \geq 0.$$

We denote by $v \geq 0$ the initial value of the project. To avoid confusion with \mathcal{P} , we denote by \mathcal{P}^g the corresponding investment problem. To ensure the existence of a solution to \mathcal{P}^g , we assume that $r > 1$. From Liptser and Shiryaev (1977, Theorems 7.12 and 9.1), the belief process (P, \mathcal{F}^V) satisfies again (3), where the innovation process (\bar{W}, \mathcal{F}^V) is a standard Wiener process given by:

$$(37) \quad d\bar{W}_t = \frac{dV_t/V_t - P_t dt}{\sigma}; \quad t \geq 0.$$

From (3)–(37), the joint value-belief process can be rewritten as:

$$(38) \quad d \begin{bmatrix} V_t^v \\ P_t^p \end{bmatrix} = \begin{bmatrix} P_t^p V_t^v \\ 0 \end{bmatrix} dt + \begin{bmatrix} \sigma V_t^v \\ P_t^p(1 - P_t^p)/\sigma \end{bmatrix} d\bar{W}_t; \quad t \geq 0.$$

We denote by G^{*g} and τ^{*g} the value function and the optimal stopping time for the unknown drift problem \mathcal{P}^g . One can then restate \mathcal{P}^g as in (6), with the sole difference that G^{*g} is now defined over $\mathbb{R}_+ \times [0, 1]$. Existence of an optimal stopping time τ^{*g} and of an investment boundary function v^{*g} follows along the same lines as in the proof of Proposition 1, bearing in mind that $r > 1$. The extension of Lemma 2 to the case of geometric Brownian motion is straightforward, and one obtains the following characterization of the value function.

COROLLARY 8. *For any $(v, p) \in \mathbb{R}_+ \times [0, 1]$,*

$$(39) \quad G^{*g}(v, p) = \sup_{\tau \in T^B} \mathbb{E}_{\mathbb{Q}} \left[e^{-r\tau} \left\{ 1 + p \left[\exp \left(\frac{B_\tau}{\sigma} - \frac{\tau}{2\sigma^2} \right) - 1 \right] \right\} \left[v \exp \left(\sigma B_\tau - \frac{\sigma^2}{2} \tau \right) - I \right] \right].$$

The analogue of (12) is:

$$(40) \quad G^{*g}(v, p) = \sup_{\tau \in T^V} \mathbb{E} \left[e^{-r\tau} \left[v \exp \left(\int_0^\tau P_t^p dt + \sigma \bar{W}_\tau - \frac{\sigma^2}{2} \tau \right) - I \right] \right]; \quad (v, p) \in \mathbb{R}_+ \times [0, 1].$$

From (39) and (40), it is easy to check that Corollary 1, 2 and 3 hold for the value function G^{*g} . Using the same arguments as those provided for Lemmas 3 and 4, it follows that the

investment boundary function v^{*g} is non-decreasing on $[0, 1]$ and left-continuous on $(0, 1]$. For any $t \geq 0$, let $M_t = \exp(\sigma \bar{W}_t - \sigma^2 t/2)$. By analogy with Lemma 5, the proof that v^{*g} is right-continuous on $[0, 1)$ requires the following auxiliary result.

LEMMA 14. *For any $(p_0, x) \in [0, 1) \times (1, \infty)$, there exists $p > p_0$ such that:*

$$(41) \quad \sup_{\tau \in T^V} \mathbb{E} \left[e^{-r\tau} M_\tau \left[x \exp \left(\int_0^\tau P_t^p dt \right) - \exp \left(\int_0^\tau P_t^{p_0} dt \right) \right] \right] = x - 1.$$

One has then the following analogue of Lemma 6.

LEMMA 15. *v^{*g} is right-continuous on $[0, 1)$.*

PROOF. Let $p_0 \in [0, 1)$, and suppose that $\lim_{p \downarrow p_0} v^{*g}(p) = v^{*g}(p_0^+) > v^{*g}(p_0)$. Fix some $v \in (v^{*g}(p_0), v^{*g}(p_0^+))$. For any $p \in (p_0, 1]$, let $\tau_{v,p}^{*g}$ be the optimal stopping time given initial conditions (v, p) . Then, by the analogue of (6) and (40),

$$(42) \quad \begin{aligned} G^{*g}(v, p) &= \mathbb{E} \left[e^{-r\tau_{v,p}^{*g}} \left[v \exp \left(\int_0^{\tau_{v,p}^{*g}} P_t^p dt + \sigma \bar{W}_{\tau_{v,p}^{*g}} - \frac{\sigma^2}{2} \tau_{v,p}^{*g} \right) - I \right] \right] \\ &\leq v^{*g}(p_0) - I \\ &\quad + v^{*g}(p_0) \mathbb{E} \left[e^{-r\tau_{v,p}^{*g}} M_{\tau_{v,p}^{*g}} \left[\frac{v}{v^{*g}(p_0)} \exp \left(\int_0^{\tau_{v,p}^{*g}} P_t^p dt \right) - \exp \left(\int_0^{\tau_{v,p}^{*g}} P_t^{p_0} dt \right) \right] \right], \end{aligned}$$

where we have used the fact that $G^{*g}(v^{*g}(p_0), p_0) = v^{*g}(p_0) - I$. Since $v/v^{*g}(p_0) > 1$, Lemma 14 implies that there exists $p > p_0$ such that:

$$\mathbb{E} \left[e^{-r\tau_{v,p}^{*g}} M_{\tau_{v,p}^{*g}} \left[\frac{v}{v^{*g}(p_0)} \exp \left(\int_0^{\tau_{v,p}^{*g}} P_t^p dt \right) - \exp \left(\int_0^{\tau_{v,p}^{*g}} P_t^{p_0} dt \right) \right] \right] \leq \frac{v}{v^{*g}(p_0)} - 1.$$

Hence, from (42), $G^{*g}(v, p) \leq v - I$ and, since the reverse inequality always hold, $G^{*g}(v, p) = v - I$, so that $v^{*g}(p) \leq v$ by definition of v^{*g} . As v^{*g} is non-decreasing, it follows that $v \geq v^{*g}(p_0^+)$. This contradicts the fact that, by assumption, $v < v^{*g}(p_0^+)$. \square

We have thus proved the following result.

THEOREM 4. *The investment boundary function $v^{*g} : [0, 1] \rightarrow \mathbb{R}_{++}$ is continuous and non-decreasing on $[0, 1]$.*

We now check the robustness of our comparison result. In the average drift problem, the value is observable and follows a geometric Brownian motion with known drift $p \in [0, 1]$ and known variance σ ,

$$(43) \quad d\hat{V}_t = p\hat{V}_t dt + \sigma\hat{V}_t dW_t; \quad t \geq 0.$$

We denote by $\hat{G}^g(\cdot, p)$ and $\hat{\tau}^g(\hat{v}^g(p))$ the value function and the optimal stopping time for the average drift problem $\hat{\mathcal{P}}$. One can restate $\hat{\mathcal{P}}^g$ as in (16), with the sole difference that $\hat{G}^g(\cdot, p)$ is now defined over \mathbb{R}_+ . The following result is standard (Dixit and Pindyck 1994, §5.2).

LEMMA 16. *For any $p \in [0, 1]$, let $\beta = 1/2 - p/\sigma^2 + \sqrt{(1/2 - p/\sigma^2)^2 + 2r/\sigma^2}$ and $\hat{v}^g(p) = \beta I/(\beta - 1)$. Then $\hat{\tau}^g(\hat{v}^g(p))$ is an optimal stopping time for $\hat{\mathcal{P}}$.*

The comparison between G^{*g} and \hat{G}^g when the value processes V and \hat{V} follow geometric Brownian motions can be reduced to the arithmetic Brownian motion case by observing that Lemmas 8 and 9 remain valid when the unknown drift in (1) can take the values $-\sigma^2/2$ and $1 - \sigma^2/2$ with respective probabilities $1 - p$ and p , and when the payoff function $\hat{\psi}$ is given by $\hat{\psi}(v) = \exp(v) - I$ for any $v \in \mathbb{R}$. The key property in that respect is that the operator \hat{Q}_δ preserves convexity. It therefore follows that:

$$\begin{aligned} & \sup_{\tau \in \mathcal{T}^V} \mathbb{E} \left[e^{-r\tau} \left[\exp \left(\ln(v) + \left(\mu - \frac{\sigma^2}{2} \right) \tau + \sigma W_\tau \right) - I \right] \right] \\ & \geq \sup_{\tau \in \mathcal{T}^W} \mathbb{E} \left[e^{-r\tau} \left[\exp \left(\ln(v) + \left(p - \frac{\sigma^2}{2} \right) \tau + \sigma W_\tau \right) - I \right] \right]; \quad (v, p) \in \mathbb{R}_+ \times [0, 1], \end{aligned}$$

and one has the following result.

THEOREM 5. *For any $(v, p) \in \mathbb{R}_+ \times [0, 1]$,*

$$G^{*g}(v, p) \geq \hat{G}^g(v, p).$$

The above discussion suggests that the only difference between the arithmetic and the geometric Brownian motion cases lies in the payoff function, which is linear in the arithmetic case, and exponential in the geometric case. This additional element of convexity makes the comparison result less surprising in the geometric than in the arithmetic case. Strict comparison results analogous to Theorem 3 and Corollary 4 are easy to derive, and are therefore omitted.

8. Concluding remarks. This paper has focused on the qualitative properties of the optimal decision to invest in a project whose value is observable but driven by a parameter that is unknown to the decision-maker ex-ante. In the case where the drift can take only two values, we have shown that the optimal investment strategy is characterized by a continuous and non-decreasing boundary in the value-belief state space. The presence of learning implies that the optimal investment strategy is path-dependent. In particular, the value of the project at the time of the investment does not necessarily coincide with its historic maximum.

We have shown that the decision-maker always benefit from being uncertain about the drift of the value process, in the sense that he prefers the option to invest in a project with unknown drift to that of investing in a project with a constant drift equal to the prior expectation of the drift in the first option. Thus one might expect the value of claims on structurally risky assets, for instance in an emerging sector in which future growth prospects are uncertain, to be higher than that of claims on assets in more traditional sectors with otherwise identical risk characteristics.

A significant point of departure with standard real option models is that the value of the option to invest is not everywhere increasing with respect to the volatility of the value process. Thus, while drift uncertainty always benefit a risk-neutral investor, non-structural uncertainty might prove harmful. As we argued, this non-monotonicity can be interpreted in terms of two countervailing effects, the real option effect and the inefficient learning effect.

Appendix.

PROOF OF LEMMA 1. For any $t \geq 0$, $P_t^0 = \int_0^t P_s^0 (1 - P_s^0) d\bar{W}_s$ satisfies:

$$\mathbb{E}[(P_t^0)^2] = \int_0^t \mathbb{E}[(P_s^0)^2 (1 - P_s^0)^2] ds \leq \int_0^t \mathbb{E}[(P_s^0)^2] ds.$$

Thus, by Gronwall's lemma, $\mathbb{E}[(P_t^0)^2] = 0$, and therefore $P_t^0 = 0$, \mathbb{P} -almost surely. Since $1 - P_t^1 = \int_0^t P_s^0 (1 - P_s^0) d\bar{W}_s$, this also implies that $P_t^1 = 1$, \mathbb{P} -almost surely. Hence (i) follows. (ii) is a direct application of Feller's test for explosions (Karatzas and Shreve 1991, Theorem 5.5.29). \square

PROOF OF PROPOSITION 1. Note first that, since $\inf\{t \geq 0 \mid V_t^v \geq I\} \in \mathcal{T}^V$, the supremum in (6) must be non-negative. Next, from (5) and the definition of g , it follows that, for any $(v, p) \in \mathbb{R} \times [0, 1]$,

$$\begin{aligned} G^*(v, p) &= \sup_{\tau \in \mathcal{T}^V} \mathbb{E} \left[e^{-r\tau} \left(v + \int_0^\tau P_t^p dt + \sigma \bar{W}_\tau - I \right) \right] \leq \sup_{\tau \in \mathcal{T}^V} \mathbb{E} [e^{-r\tau} (v + \tau + \sigma \bar{W}_\tau - I)] \\ (44) \qquad \qquad \qquad &= G^*(v, 1), \end{aligned}$$

where the last equality follows from the fact that \bar{W} is a Brownian motion under \mathcal{F}^V . In particular, G^* is well-defined. Let $\mathcal{C}^* = \{(v, p) \in \mathbb{R} \times [0, 1] \mid G^*(v, p) > g(v, p)\}$ be the continuation region for our problem, and let $\tau^* = \inf\{t \geq 0 \mid X_t^{v,p} \notin \mathcal{C}^*\}$. Since the family of random variables $\{e^{-r\tau} \bar{W}_\tau^- \mid \tau \in \mathcal{T}^V\}$ is uniformly integrable as $\sup_{\tau \in \mathcal{T}^V} \mathbb{E}[e^{-2r\tau} W_\tau^2] < \infty$, a sufficient condition for:

$$G^*(v, p) = \mathbb{E} [e^{-r\tau^*} g(X_{\tau^*}^{v,p})] = \mathbb{E} \left[e^{-r\tau^*} \left(v + \int_0^{\tau^*} P_t^p dt + \sigma \bar{W}_{\tau^*} - I \right) \right]$$

is that τ^* be \mathbb{P} -almost surely finite (Øksendal 2000, Theorem 10.1.9, and discussion p. 205). To prove this, note that $\tau^* \leq T^v(v^*(1)) = \inf\{t \geq 0 \mid V_t^v \geq v^*(1)\}$, \mathbb{P} -almost surely. Indeed, if not, then with positive \mathbb{P} -probability,

$$G^*(v^*(1), P_{T^v(v^*(1))}^p) > v^*(1) - I = G^*(v^*(1), 1),$$

which contradicts (44). Since $T^v(v^*(1)) \leq \inf\{t \geq 0 \mid v + \sigma W_t \geq v^*(1)\}$ which is \mathbb{P} -almost surely finite, (i) follows. Furthermore, $\mathcal{S}^* = \mathbb{R} \times [0, 1] \setminus \mathcal{C}^* \neq \emptyset$. Suppose now that $(v, p) \in \mathcal{S}^*$, so that $G^*(v, p) = v - I$. For any $h > 0$, discounting implies that $G^*(v+h, p) \leq G^*(v, p) + h = v+h-I$. Since the reverse inequality always holds as immediate stopping is a feasible strategy, one has $G^*(v+h, p) = v+h-I$, hence $(v+h, p) \in \mathcal{S}^*$. As $\tau^* \leq T^v(v^*(1))$, (ii) follows. \square

PROOF OF LEMMA 2. For any $t \geq 0$, let $Y_t = e^{-rt}(V_t - I)$. The following holds. (i) $\lim_{t \rightarrow \infty} Y_t = 0$, \mathbb{P} - and \mathbb{Q} -almost surely. (ii) The sequence $\{|Y_n|\}$ converges to 0 in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ as $\mathbb{E}[e^{-rn}|W_n|] \leq e^{-rn}\sqrt{2n/\pi}$ for any $n \in \mathbb{N}$. (iii) The sequence $\{H_n(\mu)|Y_n|\}$ converges to 0 in $L^1(\Omega, \mathcal{F}, \mathbb{Q})$ as $\mathbb{E}_{\mathbb{Q}}[H_n(\mu)|Y_n|] = \mathbb{E}[|Y_n|]$. (iv) Finally $\sup_{\tau \in \mathcal{T}^V} \mathbb{E}[|Y_\tau|] < \infty$ as $\sup_{\tau \in \mathcal{T}^V} \mathbb{E}[e^{-r\tau}|W_\tau|] \leq \sup_{\tau \in \mathcal{T}^W} \mathbb{E}[e^{-r\tau}|W_\tau|] < \infty$, where the first inequality follows from the inclusion $\mathcal{F}^V \subset \mathcal{F}^{\mu, W}$ and the independence of μ and W , and the second from standard optimal stopping theory. For any $\tau \in \mathcal{T}^V$ and $n \in \mathbb{N}$, one has $\mathbb{E}[Y_{\tau \wedge n}] = \mathbb{E}_{\mathbb{Q}}[H_{\tau \wedge n}(\mu)Y_{\tau \wedge n}]$, or, equivalently,

$$(45) \quad \mathbb{E}[Y_\tau 1_{\{\tau \leq n\}}] + \mathbb{E}[Y_n 1_{\{\tau > n\}}] = \mathbb{E}_{\mathbb{Q}}[H_\tau(\mu)Y_\tau 1_{\{\tau \leq n\}}] + \mathbb{E}_{\mathbb{Q}}[H_n(\mu)Y_n 1_{\{\tau > n\}}].$$

We study the convergence of each term in (45) as n goes to infinity. Since $\mathbb{E}[|Y_\tau|] < \infty$ by (iv), the dominated convergence theorem implies that the sequence $\{\mathbb{E}[Y_\tau 1_{\{\tau \leq n\}}]\}$ converges to $\mathbb{E}[Y_\tau 1_{\{\tau < \infty\}}]$, which is equal to $\mathbb{E}[Y_\tau]$ by (i). By (ii), the sequence $\{\mathbb{E}[Y_n 1_{\{\tau > n\}}]\}$ converges to 0 in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and thus the sequence $\{\mathbb{E}[Y_n 1_{\{\tau > n\}}]\}$ converges to 0. To apply the same

reasoning to the right-hand side of (45), one simply needs to prove that $\mathbb{E}_{\mathbb{Q}}[H_{\tau}(\mu)|Y_{\tau}] < \infty$. To see this, note that, by Fatou's lemma and (iv):

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[H_{\tau}(\mu)|Y_{\tau}] &= \mathbb{E}_{\mathbb{Q}}\left[\liminf_{n \rightarrow \infty} H_{\tau \wedge n}(\mu)|Y_{\tau \wedge n}\right] \leq \liminf_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[H_{\tau \wedge n}(\mu)|Y_{\tau \wedge n}] \\ &= \liminf_{n \rightarrow \infty} \mathbb{E}[|Y_{\tau \wedge n}|] = \mathbb{E}[|Y_{\tau}|] < \infty,\end{aligned}$$

where the first equality follows from (i), and the last from an argument similar to the one used to prove that the left-hand side of (45) converges to $\mathbb{E}[Y_{\tau}]$. Thus, letting n go to infinity in (45) yields $\mathbb{E}[Y_{\tau}] = \mathbb{E}_{\mathbb{Q}}[H_{\tau}(\mu)Y_{\tau}]$ for any $\tau \in \mathcal{T}^V$, which implies the result by (8). \square

PROOF OF COROLLARY 3. First, note from (6) that, for any $(v, v_0, p) \in \mathbb{R}^2 \times [0, 1]$,

$$(46) \quad |G^*(v, p) - G^*(v_0, p)| \leq |v - v_0|.$$

For any $(v, p) \in \mathbb{R} \times [0, 1]$, let $\tau_{v,p}^*$ be the optimal stopping time for problem \mathcal{P} given initial conditions (v, p) . Using (10) and the fact that the mapping $p \mapsto G^*(v, p)$ is non-decreasing by Corollary 1, simple manipulations imply that for any $(v, p, p_0) \in \mathbb{R} \times [0, 1]^2$,

$$\begin{aligned}(47) \quad \left| \frac{G^*(v, p) - G^*(v, p_0)}{p - p_0} \right| &\leq \max_{p_1 \in \{p, p_0\}} \mathbb{E}_{\mathbb{Q}} \left[e^{-r\tau_{v,p_1}^*} \left(v + \sigma B_{\tau_{v,p_1}^*} - I \right) \left[\exp \left(\frac{B_{\tau_{v,p_1}^*}}{\sigma} - \frac{\tau_{v,p_1}^*}{2\sigma^2} \right) - 1 \right] \right] \\ &\leq \sup_{\tau \in \mathcal{T}^V} \mathbb{E}_{\mathbb{Q}} \left[e^{-r\tau} (v + \sigma B_{\tau} - I) \left[\exp \left(\frac{B_{\tau}}{\sigma} - \frac{\tau}{2\sigma^2} \right) + 1 \right] \right] \\ &\leq G^*(v, 0) + G^*(v, 1),\end{aligned}$$

where the second inequality follows from the fact that both $v + \sigma B_{\tau_{v,p}^*} - I$ and $v + \sigma B_{\tau_{v,p_0}^*} - I$ must be non-negative \mathbb{P} -almost surely, and the third from (10), applied respectively to $p = 0$ and $p = 1$. Using the two uniform upper bounds (46) and (47), we obtain that for any $(v, v_0, p, p_0) \in \mathbb{R}^2 \times [0, 1]^2$,

$$\begin{aligned}|G^*(v, p) - G^*(v_0, p_0)| &\leq |G^*(v, p) - G^*(v_0, p)| + |G^*(v_0, p) - G^*(v_0, p_0)| \\ &\leq |v - v_0| + [G^*(v_0, 0) + G^*(v_0, 1)]|p - p_0|,\end{aligned}$$

which implies (i). To prove (ii), let $N_t = \exp(B_t/\sigma - t/(2\sigma^2)) - \exp(B_t/\sigma_0 - t/(2\sigma_0^2))$ for any $t \geq 0$, where (B, \mathcal{F}^B) is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q})$ independent of μ . Using (10) and proceeding as for (47), we obtain that for any $(v, p, \sigma, \sigma_0) \in \mathbb{R} \times [0, 1] \times \mathbb{R}_{++}^2$,

$$\begin{aligned}(48) \quad |G_{\sigma}^*(v, p) - G_{\sigma_0}^*(v, p)| &\leq |\sigma - \sigma_0| \max_{\sigma_1 \in \{\sigma, \sigma_0\}} \sup_{\tau \in \mathcal{T}^B} \left| \mathbb{E}_{\mathbb{Q}} \left[e^{-r\tau} \left\{ 1 + p \left[\exp \left(\frac{B_{\tau}}{\sigma_1} - \frac{\tau}{2\sigma_1^2} \right) - 1 \right] \right\} B_{\tau} \right] \right| \\ &\quad + p \max_{\sigma_1 \in \{\sigma, \sigma_0\}} \sup_{\tau \in \mathcal{T}^B} \left| \mathbb{E}_{\mathbb{Q}} [e^{-r\tau} N_{\tau} (v + \sigma_1 B_{\tau} - I)] \right|.\end{aligned}$$

Let $\sigma_1 \in \{\sigma, \sigma_0\}$. Along the lines of Subsection 3.2, one can show that there exists a unique probability measure \mathbb{P}^{σ_1} on (Ω, \mathcal{F}) such that:

$$\mathbb{P}^{\sigma_1}[A] = \mathbb{E}_{\mathbb{Q}} \left[1_A \exp \left(\frac{B_t}{\sigma_1} - \frac{t}{2\sigma_1^2} \right) \right]$$

for any $A \in \mathcal{F}_t^B$, and a direct application of Girsanov's theorem (Karatzas and Shreve 1991, Corollary 3.5.2) implies that the process $(B^{\sigma_1}, \mathcal{F}^B)$ defined by $B_t^{\sigma_1} = B_t - t/\sigma_1$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P}^{\sigma_1})$. Using the fact that $\mathcal{T}^B = \mathcal{T}^{B^{\sigma_1}}$, and an argument similar to Lemma 2, it is then easy to check that:

$$\sup_{\tau \in \mathcal{T}^B} \left| \mathbb{E}_{\mathbb{Q}} \left[e^{-r\tau} \left\{ 1 + p \left[\exp \left(\frac{B_\tau}{\sigma_1} - \frac{\tau}{2\sigma_1^2} \right) - 1 \right] \right\} B_\tau \right] \right| \leq \sup_{\tau \in \mathcal{T}^{B^{\sigma_1}}} \mathbb{E}_{\mathbb{P}^{\sigma_1}} [e^{-r\tau} |B_\tau^{\sigma_1}|] + \frac{1}{\sigma_1} \sup_{t \geq 0} e^{-rt} t.$$

From standard optimal stopping theory, the first term on the right-hand side of this inequality is finite and independent of σ_1 , and the second is bounded for σ in a neighborhood of σ_0 as $\sigma_0 > 0$. It follows that the first term on the right-hand side of (48) converges to 0 as σ converges to σ_0 . Consider now the second term on the right-hand side of (48). Proceeding as above, it is straightforward to check that:

$$\sup_{\tau \in \mathcal{T}^B} |\mathbb{E}_{\mathbb{Q}} [e^{-r\tau} N_\tau(v + \sigma_1 B_\tau - I)]| \leq 2 \max_{\sigma_2 \in \{\sigma, \sigma_0\}} \sup_{\tau \in \mathcal{T}^{B^{\sigma_2}}} \mathbb{E}_{\mathbb{P}^{\sigma_2}} \left[e^{-r\tau} \left| v + \frac{\sigma_1}{\sigma_2} \tau + \sigma_1 B_\tau^{\sigma_2} - I \right| \right],$$

which is easily seen to be finite. The dominated convergence theorem implies then that:

$$(49) \quad \lim_{n \rightarrow \infty} \sup_{\tau \in \mathcal{T}^B} |\mathbb{E}_{\mathbb{Q}} [e^{-r\tau} N_\tau(v + \sigma_1 B_\tau - I) 1_{\{\tau > n\}}]| = 0.$$

Next, for any $(\tau, n) \in \mathcal{T}^B \times \mathbb{N}$, we obtain, from Jensen's and Cauchy-Schwartz inequalities:

$$\begin{aligned} |\mathbb{E}_{\mathbb{Q}} [e^{-r\tau} N_\tau(v + \sigma_1 B_\tau - I) 1_{\{\tau \leq n\}}]| &\leq \mathbb{E}_{\mathbb{Q}} [e^{-r\tau} |N_\tau(v + \sigma_1 B_\tau - I) 1_{\{\tau \leq n\}}|] \\ (50) \quad &\leq \sqrt{\mathbb{E}_{\mathbb{Q}} [N_\tau^2 1_{\{\tau \leq n\}}] \mathbb{E}_{\mathbb{Q}} [(v + \sigma_1 B_\tau - I)^2 1_{\{\tau \leq n\}}]} \\ &\leq 2 \sqrt{\mathbb{E}_{\mathbb{Q}} [N_n^2] [(v - I)^2 + 4\sqrt{n} |v - I| \sigma_1 + 4\sigma_1^2]}, \end{aligned}$$

where the last step follows from:

$$\mathbb{E}_{\mathbb{Q}} \left[\sup_{t \leq n} B_t \right] \leq \sqrt{\mathbb{E}_{\mathbb{Q}} \left[\sup_{t \leq n} B_t^2 \right]}$$

together with Doob's inequality applied to the martingales B and N . It is easy to check from the definition of N that $N_n = \int_0^n [(1/\sigma - 1/\sigma_0) \exp(B_t/\sigma - t/(2\sigma^2)) + N_t/\sigma_0] dB_t$. From Itô's isometry and Gronwall's inequality, we get, after some straightforward computations:

$$(51) \quad \mathbb{E}_{\mathbb{Q}} [N_n^2] \leq \frac{\sigma}{\sigma_0} \sigma \left(\frac{1}{\sigma} - \frac{1}{\sigma_0} \right) \left[(\sigma + \sigma_0) (e^{n/\sigma^2} - 1) - 2\sigma_0 (e^{n/(\sigma\sigma_0)} - 1) \right] e^{n/\sigma_0^2}.$$

Let $\varepsilon > 0$. By (49), $\sup_{\tau \in \mathcal{T}^B} |\mathbb{E}_{\mathbb{Q}} [e^{-r\tau} N_\tau(v + \sigma_1 B_\tau - I) 1_{\{\tau > n\}}]| < \varepsilon/2$ for large enough n . Similarly, from (50) and (51), $\sup_{\tau \in \mathcal{T}^B} |\mathbb{E}_{\mathbb{Q}} [e^{-r\tau} N_\tau(v + \sigma_1 B_\tau - I) 1_{\{\tau \leq n\}}]| < \varepsilon/2$ for σ close enough to σ_0 . It follows that the second term on the right-hand side of (48) converges to 0 as σ converges to σ_0 , which concludes the proof of (ii). \square

PROOF OF LEMMA 5. Suppose first that $p_0 \in (0, 1)$, and let $p \in (p_0, 1]$. Then, from (11), the process $P^p - P^{p_0}$ can be written as $(p - p_0)f(H(1), p, p_0)$, where $H(1)$ is defined as in (9), and for any $h \in \mathbb{R}_+$,

$$(52) \quad f(h, p, p_0) = \frac{h}{pp_0(h - 1)^2 + (p + p_0)(h - 1) + 1}.$$

It is easy to check from (52) that the mapping $h \mapsto f(h, p, p_0)$ reaches a maximum on \mathbb{R}_+ at $h(p, p_0) = \sqrt{(1-p)(1-p_0)/pp_0}$, and that the mapping $p \mapsto f(h(p, p_0), p, p_0)$ is bounded above by some positive constant $C(p_0)$ in a neighborhood of p_0 . Thus,

$$(53) \quad \sup_{t \geq 0} |P_t^p - P_t^{p_0}| \leq C(p_0)|p - p_0|$$

for all p in a neighborhood of p_0 , \mathbb{P} -almost surely. It follows from (53) that:

$$\sup_{\tau \in \mathcal{T}^V} \mathbb{E} \left[e^{-r\tau} \left[x + \int_0^\tau (P_t^p - P_t^{p_0}) dt \right] \right] \leq \sup_{t \geq 0} e^{-rt} [x + C(p_0)(p - p_0)t].$$

The mapping $t \mapsto e^{-rt} [x + C(p_0)(p - p_0)t]$ is decreasing on \mathbb{R}_+ if $C(p_0)(p - p_0) \leq rx$. Since the supremum in (13) is greater or equal than x , the result follows. Suppose now that $p_0 = 0$. Then, by Lemma 1, $P_t^{p_0} = 0$ \mathbb{P} -almost surely, and, for any $\tau \in \mathcal{T}^V$,

$$(54) \quad \mathbb{E} \left[e^{-r\tau} \left(x + \int_0^\tau P_t^p dt \right) \right] \leq \mathbb{E} \left[e^{-r\tau} x + \int_0^\tau e^{-rt} P_t^p dt \right] = \frac{p}{r} + \mathbb{E} \left[e^{-r\tau} x - \int_\tau^\infty e^{-rt} P_t^p dt \right],$$

where the equality follows from the monotone convergence theorem. Next,

$$\begin{aligned} \mathbb{E} \left[\int_\tau^\infty e^{-rt} P_t^p dt \right] &= \mathbb{E} \left[\mathbb{E} \left[\int_\tau^\infty e^{-rt} P_t^p dt \mid \mathcal{F}_\tau^V \right] \right] \\ &= \mathbb{E} \left[e^{-r\tau} \mathbb{E} \left[\int_0^\infty e^{-rt} P_{t+\tau}^p dt \mid \mathcal{F}_\tau^V \right] \right] \\ &= \mathbb{E} \left[e^{-r\tau} \int_0^\infty e^{-rt} \mathbb{E} [P_{t+\tau}^p \mid \mathcal{F}_\tau^V] dt \right] \\ &= \mathbb{E} \left[e^{-r\tau} \frac{P_\tau^p}{r} \right], \end{aligned}$$

where the third equality follows from the monotone convergence theorem and the fourth from the strong Markov property together with the fact that P^p is a martingale. From (54), we thus need only to prove that there exists some $p > 0$ such that:

$$(55) \quad G^\circ(p) = \sup_{\tau \in \mathcal{T}^V} \mathbb{E} \left[e^{-r\tau} \left(x - \frac{P_\tau^p}{r} \right) \right] = x - \frac{p}{r}.$$

If $rx \geq 1$, $G^\circ(p) = x - p/r$ since $x - P^p/r$ is then a positive martingale. If $rx < 1$, a standard computation (see for instance Bolton and Harris 1999) yields that a solution to (55) is given by $\tau^\circ = \inf\{t \geq 0 \mid P_t^p \leq p^\circ\}$, where $p^\circ = (\gamma - 1)rx/(\gamma - 2rx + 1) > 0$ with $\gamma = \sqrt{1 + 8r\sigma^2}$. Hence $G^\circ(p) = x - p/r$ if $p \leq p^\circ$, which implies the result. \square

PROOF OF PROPOSITION 3. For any $(v, p) \in \mathbb{R} \times [0, 1]$ and for any $\bar{v} \geq v$, one gets, using (22) and the definitions of \hat{v} and f :

$$\frac{\partial \Gamma(v, p, \bar{v})}{\partial \bar{v}} = \frac{pf(1)}{\sigma^2} \exp\left(\frac{v - \bar{v}}{\sigma^2} f(1)\right) [\hat{v}(1) - \bar{v}] + \frac{(1-p)f(0)}{\sigma^2} \exp\left(\frac{v - \bar{v}}{\sigma^2} f(0)\right) [\hat{v}(0) - \bar{v}].$$

Thus $\operatorname{argmax}_{\bar{v} \geq v} \Gamma(v, p, \bar{v}) = \{v\}$ whenever $v \geq \hat{v}(1)$, and $\operatorname{argmax}_{\bar{v} \geq v} \Gamma(v, p, \bar{v}) \subset [\hat{v}(0), \hat{v}(1))$ otherwise. From now on, we focus on the latter case. Let us first show that $\bar{v} \mapsto \Gamma(v, p, \bar{v})$ is

quasi-concave on $[v, \hat{v}(1)]$. If $\partial\Gamma(v, p, \bar{v})/\partial\bar{v} < 0$ whenever $\bar{v} \in [v, \hat{v}(1)]$, the result is immediate and $\operatorname{argmax}_{\bar{v} \geq v} \Gamma(v, p, \bar{v}) = \{v\}$. Otherwise, let $\bar{v} \in [v, \hat{v}(1)]$ such that $\partial\Gamma(v, p, \bar{v})/\partial\bar{v} = 0$. Then, one has:

$$(56) \quad \begin{aligned} \frac{\partial^2 \Gamma(v, p, \bar{v})}{\partial \bar{v}^2} &\propto [\hat{v}(1) - \bar{v}][\bar{v} - \hat{v}(0)] \frac{f(0) - f(1)}{\sigma^2} - [\hat{v}(1) - \hat{v}(0)] \\ &\leq [\hat{v}(1) - \hat{v}(0)] \left\{ \frac{[\hat{v}(1) - \hat{v}(0)][f(0) - f(1)]}{4\sigma^2} - 1 \right\}. \end{aligned}$$

A direct computation reveals that the right-hand side of (56) is negative for all $(r, \sigma) \in \mathbb{R}_{++}^2$, which implies the strict quasi-concavity of $\bar{v} \mapsto \Gamma(v, p, \bar{v})$ on $[v, \hat{v}(1)]$. For any $v \in [\tilde{v}(p), \hat{v}(1)]$,

$$\left. \frac{\partial \Gamma(v, p, \bar{v})}{\partial \bar{v}} \right|_{\bar{v}=v} = 1 + \frac{I - v}{\tilde{v}(p) - I} < 0,$$

so that $\partial\Gamma(v, p, \bar{v})/\partial\bar{v} < 0$ for any $\bar{v} \in [v, \hat{v}(1)]$ by strict quasi-concavity of $\bar{v} \mapsto \Gamma(v, p, \bar{v})$ on $[v, \hat{v}(1)]$. Therefore $\operatorname{argmax}_{\bar{v} \geq v} \Gamma(v, p, \bar{v}) = \{v\}$ in that case as well. \square

PROOF OF LEMMA 11. Pointwise convergence follows immediately from (17) and (23) together with the fact that $\lim_{\sigma \rightarrow 0} \sigma^2/f_\sigma(p) = p/r$ by L'Hôpital rule. Next, for any $p \in [0, 1]$, the quantity $\sigma^2/f_\sigma(p)$ is an increasing function of σ . Since \hat{G}_0 is continuous and the mapping $x \mapsto \exp(-1 + (v - I)/x)$ is increasing on $[v - I, \infty)$ for any $v \in \mathbb{R}$, the result follows immediately from Dini's theorem. \square

PROOF OF LEMMA 12. From (6) and (25), we have, for any $(v, p) \in \mathbb{R} \times [0, 1]$,

$$\left| G_\sigma^*(v, p) - \tilde{G}_\sigma(v, p) \right| \leq \sigma \sup_{\tau \in T^{V_\sigma}} \mathbb{E}[e^{-r\tau} |W_\tau|] \leq \sigma \sup_{\tau \in T^W} \mathbb{E}[e^{-r\tau} |W_\tau|],$$

where the second inequality follows from the inclusion $\mathcal{F}^{V_\sigma} \subset \mathcal{F}^{\mu, W}$ and the independence of μ and W . Since $\sup_{\tau \in T^W} \mathbb{E}[e^{-r\tau} |W_\tau|] < \infty$ from standard optimal stopping theory, the result follows. \square

PROOF OF LEMMA 13. Note first that, for any $\sigma > 0$ and for any prior belief $p \in [0, 1]$ that $\mu = 1$, we have, from (25) and the inclusion $\mathcal{F}^{V_\sigma} \subset \mathcal{F}^{\mu, W}$,

$$\tilde{G}_\sigma(v, p) \leq \sup_{\tau \in T^{\mu, W}} \mathbb{E}[e^{-r\tau} (v + \mu\tau - I)] = p\hat{G}_0(v, 1) + (1 - p)\hat{G}_0(v, 0).$$

It follows that $\limsup_{\sigma \rightarrow 0} \tilde{G}_\sigma(v, p) \leq p\hat{G}_0(v, 1) + (1 - p)\hat{G}_0(v, 0)$ for any $(v, p) \in \mathbb{R} \times [0, 1]$. To prove the converse, consider the following strategy in $\hat{\mathcal{P}}$. First, wait for a deterministic time amount $\varepsilon \in (0, \hat{t}_0(v, p))$. Next, delay further investment by $\hat{t}_0(v + \varepsilon P_\varepsilon^p, P_\varepsilon^p)$, that is, the amount of time that is optimal in $\hat{\mathcal{P}}$ in the state $(v + \varepsilon P_\varepsilon^p, P_\varepsilon^p)$. We have:

$$(57) \quad \begin{aligned} \tilde{G}_\sigma(v, p) &\geq \mathbb{E} \left[e^{-r[\varepsilon + \hat{t}_0(v + \varepsilon P_\varepsilon^p, P_\varepsilon^p)]} \{v + \mu[\varepsilon + \hat{t}_0(v + \varepsilon P_\varepsilon^p, P_\varepsilon^p)] - I\} \right] \\ &= \mathbb{E} \left[e^{-r\varepsilon} \mathbb{E} \left[e^{-r\hat{t}_0(v + \varepsilon P_\varepsilon^p, P_\varepsilon^p)} [v + \mu\varepsilon + \mu\hat{t}_0(v + \varepsilon P_\varepsilon^p, P_\varepsilon^p) - I] \mid \mathcal{F}_\varepsilon^{V_\sigma} \right] \right] \\ &= \mathbb{E} \left[e^{-r\varepsilon} \hat{G}_0(v + \varepsilon P_\varepsilon^p, P_\varepsilon^p) \right], \end{aligned}$$

where the last equality follows from the definition of \hat{G}_0 and the fact that $\mathbb{E}[\mu \mid \mathcal{F}_\varepsilon^{V_\sigma}] = P_\varepsilon^p$. Rewriting (11) and making the dependence of P^p on σ explicit, we obtain:

$$P_\varepsilon^p(\sigma) = \frac{p \exp\left(\frac{\sigma W_\varepsilon + (\mu - \frac{1}{2})\varepsilon}{\sigma^2}\right)}{p \exp\left(\frac{\sigma W_\varepsilon + (\mu - \frac{1}{2})\varepsilon}{\sigma^2}\right) + 1 - p},$$

from which we get that $\lim_{\sigma \rightarrow 0} P_\varepsilon^p(\sigma) = \mu$, \mathbb{P} -almost surely. As $\hat{G}_0(v + \varepsilon P_\varepsilon^p(\sigma), P_\varepsilon^p(\sigma))$ is positive and bounded above by $\hat{G}_0(v + \varepsilon, 1)$ for any $\sigma > 0$, it follows from (57) and the dominated convergence theorem that:

$$\liminf_{\sigma \rightarrow 0} \tilde{G}_\sigma(v, p) \geq \mathbb{E}\left[e^{-r\varepsilon} \hat{G}_0(v + \mu\varepsilon, \mu)\right] = e^{-r\varepsilon} \left[p\hat{G}_0(v + \varepsilon, 1) + (1-p)\hat{G}_0(v, 0)\right].$$

Since \hat{G}_0 is continuous, we get that $\liminf_{\sigma \rightarrow 0} \tilde{G}_\sigma(v, p) \geq p\hat{G}_0(v, 1) + (1-p)\hat{G}_0(v, 0)$ by letting ε go to 0. Since the reverse inequality holds for the lim sup, pointwise convergence follows. To prove that this convergence is uniform, note first that:

$$\left\{x \in \mathbb{R} \mid \int_{-\varepsilon}^{\varepsilon} \frac{\sigma^2 dy}{(x+y)^2(1-x-y)^2} = \infty \text{ for any } \varepsilon > 0\right\} = \{0, 1\},$$

so that, by the Engelbert–Schmidt criterion (Karatzas and Shreve 1991, Theorem 5.5.7), the stochastic differential equation (3) has a unique solution $P^p(\sigma)$ in the sense of probability law for any initial condition $p \in [0, 1]$. Thus, from the time-change theorem for diffusion processes (Øksendal 2000, Theorem 8.5.7), $P_t^p(\sigma)$ coincides in law with $P_{t/\sigma^2}^p(1)$ for any $t \geq 0$ and $\sigma > 0$. It follows that:

$$\begin{aligned} \mathbb{E}\left[e^{-r\varepsilon} \hat{G}_0(v + \varepsilon P_\varepsilon^p(\tilde{\sigma}), P_\varepsilon^p(\tilde{\sigma}))\right] &= \mathbb{E}\left[e^{-r\varepsilon} \hat{G}_0\left(v + \varepsilon P_{\varepsilon/\tilde{\sigma}^2}^p(1), P_{\varepsilon/\tilde{\sigma}^2}^p(1)\right)\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[e^{-r\varepsilon} \hat{G}_0\left(v + \varepsilon P_{\varepsilon/\tilde{\sigma}^2}^p(1), P_{\varepsilon/\tilde{\sigma}^2}^p(1)\right) \mid \mathcal{F}_{\varepsilon/\sigma^2}^{V_1}\right]\right] \\ &\geq \mathbb{E}\left[e^{-r\varepsilon} \hat{G}_0\left(v + \varepsilon P_{\varepsilon/\sigma^2}^p(1), P_{\varepsilon/\sigma^2}^p(1)\right)\right] \\ &= \mathbb{E}\left[e^{-r\varepsilon} \hat{G}_0(v + \varepsilon P_\varepsilon^p(\sigma), P_\varepsilon^p(\sigma))\right], \end{aligned}$$

for any $\varepsilon \in (0, \hat{t}_0(v, p))$ and $\sigma > \tilde{\sigma} > 0$, where the first and last equalities follow from the above time-change argument, and the inequality from the fact that $P^p(1)$ is a martingale and \hat{G}_0 is convex as the supremum of linear functions of (v, p) , together with Jensen's inequality. Hence, the mapping $\sigma \mapsto \mathbb{E}[e^{-r\varepsilon} \hat{G}_0(v + \varepsilon P_\varepsilon^p(\sigma), P_\varepsilon^p(\sigma))]$ is decreasing. Since the mapping $(v, p) \mapsto \mathbb{E}[e^{-r\varepsilon} \hat{G}_0(v + \mu\varepsilon, \mu)] = p\hat{G}_0(v, 1) + (1-p)e^{-r\varepsilon} \hat{G}_0(v, 0)$ is continuous, it follows from Dini's theorem that:

$$(58) \quad \lim_{\sigma \rightarrow 0} \mathbb{E}\left[e^{-r\varepsilon} \hat{G}_0(v + \varepsilon P_\varepsilon^p(\sigma), P_\varepsilon^p(\sigma))\right] = \mathbb{E}\left[e^{-r\varepsilon} \hat{G}_0(v + \mu\varepsilon, \mu)\right]$$

uniformly on every compact set $K \subset \mathcal{D}$. Note that this holds for every $\varepsilon \in (0, \hat{t}_0(v, p))$. As ε goes to 0, the right-hand side of (58) converges uniformly to $p\hat{G}_0(v, 1) + (1-p)\hat{G}_0(v, 0)$ on every compact set $K \subset \mathcal{D}$. The result follows. \square

PROOF OF PROPOSITION 4. Following the same steps as in the proof of Proposition 1, it is easy to see that G^* is well-defined, with (44) replaced by:

$$\begin{aligned} G^*(v, p) &= \sup_{\tau \in \mathcal{T}^V} \mathbb{E} \left[e^{-r\tau} \left(v + \int_0^\tau P_t^p dt + \sigma \bar{W}_\tau - I \right) \right] \leq \sup_{\tau \in \mathcal{T}^V} \mathbb{E} [e^{-r\tau} (v + m^M \tau + \sigma \bar{W}_\tau - I)] \\ (59) \qquad \qquad \qquad &= G^*(v, e^M), \end{aligned}$$

where $e^M = (0, \dots, 0, 1) \in \Delta$. Let $\mathcal{C}^* = \{(v, p) \in \mathbb{R} \times \Delta \mid G^*(v, p) > g(v, p)\}$ be the continuation region for our problem, and let $\tau^* = \inf\{t \geq 0 \mid X_t^{v,p} \notin \mathcal{C}^*\}$. From optimal stopping theory, the process $\{e^{-r(t \wedge \tau^*)} G^*(V_{t \wedge \tau^*}^v, P_{t \wedge \tau^*}^p); t \geq 0\}$ is a martingale under \mathcal{F}^V (El Karoui 1981). Hence, for any $n \in \mathbb{N}$,

$$\begin{aligned} G^*(v, p) &= \mathbb{E} \left[e^{-r(n \wedge \tau^*)} G^*(V_{n \wedge \tau^*}^v, P_{n \wedge \tau^*}^p) \right] \\ (60) \qquad \qquad &= \mathbb{E} \left[e^{-r\tau^*} G^*(V_{\tau^*}^v, P_{\tau^*}^p) 1_{\{\tau^* \leq n\}} \right] + \mathbb{E} \left[e^{-rn} G^*(V_n^v, P_n^p) 1_{\{\tau^* > n\}} \right] \\ &= \mathbb{E} \left[e^{-r\tau^*} (V_{\tau^*}^v - I) 1_{\{\tau^* \leq n\}} \right] + \mathbb{E} \left[e^{-rn} G^*(V_n^v, P_n^p) 1_{\{\tau^* > n\}} \right]. \end{aligned}$$

Since $e^{-r\tau^*} (V_{\tau^*}^v - I) \geq 0$ and $e^{-r\tau^*} (V_{\tau^*}^v - I) = 0$ \mathbb{P} -almost surely on $\{\tau^* = \infty\}$, the first term on the last line of (60) converges to $\mathbb{E} [e^{-r\tau^*} (V_{\tau^*}^v - I)]$ as n goes to infinity. By (59), the second term is smaller than $\mathbb{E} [e^{-rn} G^*(V_n^v, e^M)]$. An explicit computation similar to (17) reveals that this is less than $\mathbb{E} [e^{-rn} \max\{V_n^v, C\}]$ for some positive constant C , and this in turns converges to 0 as n goes to infinity. Thus we have $G^*(v, p) = \mathbb{E} [e^{-r\tau^*} (V_{\tau^*}^v - I)]$, and τ^* is an optimal stopping time, which implies (i). (ii) then follows along the same lines as in the proof of Proposition 1. \square

PROOF OF LEMMA 14. Suppose first that $p_0 \in (0, 1)$, and let $p \in (p_0, 1]$. Proceeding as in Subsection 3.2, one can show that there exists a unique probability measure $\tilde{\mathbb{P}}$ on (Ω, \mathcal{F}) such that $\tilde{\mathbb{P}}[A] = \mathbb{E}[1_A M_t]$ for any $A \in \mathcal{F}_t^V$. Using this change of measure together with $r > 1$, $x > 1$ and (53), one obtains:

$$\begin{aligned} &\sup_{\tau \in \mathcal{T}^V} \mathbb{E} \left[e^{-r\tau} M_\tau \left[x \exp \left(\int_0^\tau P_t^p dt \right) - \exp \left(\int_0^\tau P_t^{p_0} dt \right) \right] \right] \\ &\leq \sup_{\tau \in \mathcal{T}^V} \mathbb{E} \left[e^{-(r-1)\tau} M_\tau \left[x \exp \left(\int_0^\tau P_t^p dt - \int_0^\tau P_t^{p_0} dt \right) - 1 \right] \right] \\ &\leq \sup_{t \geq 0} e^{-(r-1)t} [x \exp(C(p_0)(p - p_0)t) - 1]. \end{aligned}$$

The mapping $t \mapsto e^{-(r-1)t} [x \exp(C(p_0)(p - p_0)t) - 1]$ is decreasing on \mathbb{R}_+ if $C(p_0)(p - p_0) \leq (x-1)(r-1)/x$. Since the supremum in (41) is greater or equal than $x-1$, the result follows. Suppose now that $p_0 = 0$. Then, by Lemma 1, $P_t^{p_0} = 0$ \mathbb{P} -almost surely, and, for any $\tau \in \mathcal{T}^V$,

$$\begin{aligned} &\mathbb{E} \left[e^{-r\tau} M_\tau \left[x \exp \left(\int_0^\tau P_t^p dt \right) - 1 \right] \right] = \mathbb{E}_{\tilde{\mathbb{P}}} \left[e^{-r\tau} \left[x \exp \left(\int_0^\tau P_t^p dt \right) - 1 \right] \right] \\ (61) \qquad \qquad \qquad &\leq \mathbb{E}_{\tilde{\mathbb{P}}} \left[e^{-\rho\tau} \left(x - 1 + x \int_0^\tau P_t^p dt \right) \right], \end{aligned}$$

where $\rho = r - 1 > 0$ and the inequality follows from the convexity of the exponential function together with the fact that $P^p \leq 1$. In turn, the right-hand side of (61) is bounded above by:

$$\begin{aligned} \mathbb{E}_{\tilde{\mathbb{P}}} \left[e^{-\rho\tau} (x - 1) + x \int_0^\tau e^{-\rho t} P_t^p dt \right] &= x \mathbb{E}_{\tilde{\mathbb{P}}} \left[\int_0^\infty e^{-\rho t} P_t^p dt \right] \\ &\quad + \mathbb{E}_{\tilde{\mathbb{P}}} \left[e^{-\rho\tau} (x - 1) - x \int_\tau^\infty e^{-\rho t} P_t^p dt \right], \end{aligned}$$

where the equality follows from the monotone convergence theorem. Note that, as $P^p \in [0, 1]$, the expectation $\mathbb{E}_{\tilde{\mathbb{P}}} [\int_0^\infty e^{-\rho t} P_t^p dt]$ is well-defined and finite. Next,

$$\begin{aligned} \mathbb{E}_{\tilde{\mathbb{P}}} \left[\int_\tau^\infty e^{-\rho t} P_t^p dt \right] &= \mathbb{E} \left[\mathbb{E}_{\tilde{\mathbb{P}}} \left[\int_\tau^\infty e^{-\rho t} P_t^p dt \mid \mathcal{F}_\tau^V \right] \right] \\ &= \mathbb{E}_{\tilde{\mathbb{P}}} \left[e^{-\rho\tau} \mathbb{E}_{\tilde{\mathbb{P}}} \left[\int_0^\infty e^{-\rho t} P_{t+\tau}^p dt \mid \mathcal{F}_\tau^V \right] \right] \\ &= \mathbb{E}_{\tilde{\mathbb{P}}} \left[e^{-\rho\tau} \int_0^\infty e^{-\rho t} \mathbb{E}_{\tilde{\mathbb{P}}} [P_{t+\tau}^p \mid \mathcal{F}_\tau^V] dt \right] \\ &= \mathbb{E}_{\tilde{\mathbb{P}}} \left[e^{-\rho\tau} \int_0^\infty e^{-\rho t} f(t, P_\tau^p) dt \right], \end{aligned}$$

where the third inequality follows from the monotone convergence theorem and the fourth from the strong Markov property, with the notation $f(t, p) = \mathbb{E}_{\tilde{\mathbb{P}}} [P_t^p]$. From (61), we thus need only to prove that there exists some $p > 0$ such that:

$$(62) \quad G^\bullet(p) = \sup_{\tau \in \mathcal{T}^V} \mathbb{E}_{\tilde{\mathbb{P}}} \left[e^{-\rho\tau} \left(x - 1 - x \int_0^\infty e^{-\rho t} f(t, P_\tau^p) dt \right) \right] = x - 1 - x \mathbb{E}_{\tilde{\mathbb{P}}} \left[\int_0^\infty e^{-\rho t} P_t^p dt \right].$$

Note that (62) is a Markovian optimal stopping problem. From optimal stopping theory, the process $\{e^{-\rho(t \wedge \tau^\bullet)} G^\bullet(P_{t \wedge \tau^\bullet}^p); t \geq 0\}$ is a martingale under \mathcal{F}^V , where by definition $\tau^\bullet = \inf\{t \geq 0 \mid G^\bullet(P_t^p) = x - 1 - x \int_0^\infty e^{-\rho t} f(t, P_t^p) dt\}$ (El Karoui 1981). Using Feller's test for explosions (Karatzas and Shreve 1991, Theorem 5.5.29), it is easy to check that for any $p \in (0, 1)$, $\inf\{t \geq 0 \mid P_t^p \notin (0, 1)\} = \infty$, $\tilde{\mathbb{P}}$ -almost surely. Suppose that $\tau^\bullet = \infty$, or equivalently, that the optimal stopping region for (62) is reduced to $\{0\}$, which is not attainable in finite time. Then the process $\{e^{-\rho t} G^\bullet(P_t^p); t \geq 0\}$ is a martingale. Thus for any $p > 0$ and $t \geq 0$, $G^\bullet(p) = \mathbb{E}_{\tilde{\mathbb{P}}} [e^{-\rho t} G^\bullet(P_t^p)] \leq e^{-\rho t} (x - 1)$ and therefore $G^\bullet(p) = 0$ for any $p > 0$. However, for any such p ,

$$(63) \quad G^\bullet(p) \geq x - 1 - x \int_0^\infty e^{-\rho t} f(t, p) dt = x - 1 - x \mathbb{E}_{\tilde{\mathbb{P}}} \left[\int_0^\infty e^{-\rho t} P_t^p dt \right].$$

We shall get a contradiction by showing that $\lim_{p \downarrow 0} \mathbb{E}_{\tilde{\mathbb{P}}} [\int_0^\infty e^{-\rho t} P_t^p dt] = 0$. A standard application of Doob's inequality and Gronwall's lemma imply that for any $n \in \mathbb{N}$, there exists a positive constant C_n such that:

$$\mathbb{E}_{\tilde{\mathbb{P}}} \left[\sup_{t \leq n} (P_t^p)^2 \right] \leq C_n p^2$$

for any $p > 0$. Using Cauchy-Schwartz inequality, it follows that:

$$\begin{aligned}
(64) \quad \mathbb{E}_{\hat{\mathbb{P}}} \left[\int_0^\infty e^{-\rho t} P_t^p dt \right] &= \mathbb{E}_{\hat{\mathbb{P}}} \left[\int_0^n e^{-\rho t} P_t^p dt \right] + \mathbb{E}_{\hat{\mathbb{P}}} \left[\int_n^\infty e^{-\rho t} P_t^p dt \right] \\
&\leq \sqrt{\mathbb{E}_{\hat{\mathbb{P}}} \left[\sup_{t \leq n} (P_t^p)^2 \right] \int_0^n e^{-2\rho t} dt} + \frac{e^{-\rho n}}{\rho} \\
&\leq \sqrt{\frac{C_n}{2\rho}} p + \frac{e^{-\rho n}}{\rho}
\end{aligned}$$

for any $p > 0$. Since $x > 1$, there exists $\varepsilon > 0$ such that $x(1 - \varepsilon) - 1 > 0$. Choosing n such that $e^{-\rho n}/\rho \leq \varepsilon/2$ and $p > 0$ such that $\sqrt{C_n/2\rho} p \leq \varepsilon/2$, it follows from (63) and (64) that $G^\bullet(p) \geq x(1 - \varepsilon) - 1 > 0$, a contradiction. It thus follows that the optimal stopping region for (62) is not reduced to $\{0\}$ and that there exists $p > 0$ such that (62) holds. Hence the result. \square

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